

# Low-temperature large-distance asymptotics of the transversal two-point functions of the XXZ chain

Maxime Dugave<sup>\*</sup>, Frank Göhmann<sup>†</sup>

Fachbereich C – Physik, Bergische Universität Wuppertal,  
42097 Wuppertal, Germany

Karol K. Kozłowski<sup>‡</sup>

IMB, UMR 5584 du CNRS, Université de Bourgogne, France

## Abstract

We derive the low-temperature large-distance asymptotics of the transversal two-point functions of the XXZ chain by summing up the asymptotically dominant terms of their expansion into form factors of the quantum transfer matrix. Our asymptotic formulae are numerically efficient and match well with known results for vanishing magnetic field and for short distances and magnetic fields below the saturation field.

*PACS: 05.30.-d, 75.10.Pq*

---

<sup>\*</sup>e-mail: [dugave@uni-wuppertal.de](mailto:dugave@uni-wuppertal.de)

<sup>†</sup>e-mail: [goehmann@uni-wuppertal.de](mailto:goehmann@uni-wuppertal.de)

<sup>‡</sup>e-mail: [karol.kozlowski@u-bourgogne.fr](mailto:karol.kozlowski@u-bourgogne.fr)

## 1 Introduction

Many-body quantum systems undergoing second order phase transitions renormalize to conformally invariant quantum field theories at their critical points. The form of their two-point functions at critical points is therefore determined by the conformal group [36]. These statements, which are at the heart of our understanding of many-body quantum statistical systems are hard to verify directly by analytic means, even for integrable models. For the paradigmatic spin-1/2 anisotropic Heisenberg chain a direct calculation of the large-distance asymptotics of its two-point ground state correlation functions was carried out only rather recently by two different methods [17, 20].

The anisotropic Heisenberg chain, or XXZ chain, is a model of spins interacting with their nearest neighbours by a linear combination of exchange and Ising interactions. For local spins 1/2 its Hamiltonian

$$H_L = J \sum_{j=-L+1}^L \left( \sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta (\sigma_{j-1}^z \sigma_j^z - 1) \right) \quad (1)$$

is integrable. Here the  $\sigma_j^\alpha$ ,  $\alpha = x, y, z$ , are the Pauli matrices embedded into  $\text{End}(\mathbb{C}_2^{\otimes 2L})$  and periodic boundary conditions,  $\sigma_{-L}^\alpha = \sigma_L^\alpha$ , are implied.  $J$  stands for the exchange energy and  $\Delta$  for the anisotropy parameter.

The first of the two above mentioned methods [17] is based on the asymptotic analysis of a multiple integral representation of a generating function of the longitudinal two-point functions [14, 22], while the second one [20] relies on the analysis of form factors [18, 21, 23] (matrix elements of local operators between the ground state and excited states of the Hamiltonian). A major difficulty that had to be overcome in the form factor approach comes from the fact that the individual form factors of a system at a critical point vanish in the thermodynamic limit. A possible way to cope with this problem is to keep control over the finite-size scaling behaviour of the form factors and to sum up the leading finite-size contributions to the form factor expansion before taking the thermodynamic limit. It was observed in [20] that such type of summation can be performed by means of a combinatorial formula that arose in the context of the representation theory of the infinite symmetric group [16].

In our previous work [9] we combined the form factor approach with the quantum transfer matrix approach to the thermodynamics of integrable models [25, 41, 42]. We calculated form factors of the quantum transfer matrix that appear in the form factor expansion of the thermal two-point correlation functions of the XXZ chain. We like to call these form factors thermal form factors. For finite temperatures the form factor expansion of the two-point functions is a large-distance asymptotic series expansion. Each term in the series consists of an amplitude times a factor describing exponential decay. The latter is characterized by a correlation length determined by the ratio of an eigenvalue belonging to an excited state of the quantum transfer matrix and the dominant eigenvalue. The amplitudes are products of two thermal form factors. For sufficiently high temperatures a few leading terms in the form factor expansion are expected to determine the correlation functions accurately. We plan to study this intermediate and high-temperature regime in a separate work.

Here we continue the exploration of the low-temperature regime, begun in [9], with the analysis of the transversal correlation functions. In [9] we studied the low-temperature behaviour of the longitudinal correlation functions in the antiferromagnetic

critical regime for  $|\Delta| < 1$ . In this regime the gap between the dominant eigenvalue and the lowest excited-state eigenvalues of the quantum transfer matrix closes in the zero-temperature limit  $T \rightarrow 0$ , and infinitely many correlation lengths diverge. As a consequence of the overall finiteness of the correlation functions the corresponding amplitudes and form factors must necessarily vanish. The amplitudes decay as powers of  $T$  with generally non-integer exponents whose values depend on  $\Delta$  and on the magnetic field in  $z$ -direction  $h$ . Like in case of the usual transfer matrix these ‘critical’ amplitudes and correlation lengths can be classified by particle-hole excitations above two Fermi points. The corresponding contributions to the form factor series can be summed up by means of the same summation formula as in case of the ordinary transfer matrix [20].

We observed this in our analysis of the low-temperature large-distance asymptotics of the longitudinal two-point functions [9]. In this work we carry out a similar low-temperature analysis of the transversal two-point functions. We take the opportunity to close some gaps in the arguments of our previous work. In particular, we provide a careful and more complete discussion of the low-temperature behaviour of the solutions of the non-linear integral equations [24, 25] that are fundamental for the quantum transfer matrix approach to the thermodynamics of integrable models. We also show that the final formulae for the asymptotics of the two-point functions are efficient. Our expressions for the leading amplitudes can easily be evaluated numerically. Our formulae hold for any finite magnetic field. In the limit of vanishing field we compare them with the amplitudes for  $h = 0$  obtained by Lukyanov [30, 31] using field theoretical methods. We also find amazing agreement, when we compare the low-temperature magnetic field dependence of the third neighbour correlation functions following from our large-distance asymptotic expansion with the known exact results [1].

## 2 Form factor expansion of two-point functions

The thermal expectation value of an operator  $X$  acting on the space of states of the XXZ chain is defined as

$$\langle X \rangle = \lim_{L \rightarrow \infty} \frac{\text{Tr}\{e^{-H_L/T + hS^z/T} X\}}{\text{Tr}\{e^{-H_L/T + hS^z/T}\}}, \quad (2)$$

where

$$S^z = \frac{1}{2} \sum_{j=-L+1}^L \sigma_j^z \quad (3)$$

is the conserved  $z$ -component of the total spin.

In our previous work [9] we considered the longitudinal and transversal two-point functions  $\langle \sigma_1^z \sigma_{m+1}^z \rangle$  and  $\langle \sigma_1^- \sigma_{m+1}^+ \rangle$ . We studied the longitudinal two-point function by means of a generating function. For the transversal two-point function no convenient generating function is known. We considered a direct form-factor expansion of the form

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle = \sum_{n=1}^{\infty} A_n^{-+} e^{-m/\xi_n}. \quad (4)$$

Every term in this series is characterized by a correlation length  $\xi_n$  and by an amplitude  $A_n^{-+}$ .

Correlation lengths had been studied in various contexts before [24, 26, 35, 37, 43]. They are related to ratios  $\rho_n$  of quantum transfer matrix eigenvalues [41] in the so-called Trotter limit,

$$e^{-1/\xi_n} = \rho_n. \quad (5)$$

Here  $\rho_n = \rho_n(0|0)$  and

$$\rho_n(\lambda|\alpha) = q^{s+\alpha} \exp \left\{ \int_{\mathcal{C}_n} \frac{d\mu}{2\pi i} e(\mu - \lambda) \ln \left( \frac{1 + a_n(\mu|\alpha)}{1 + a_0(\mu)} \right) \right\}. \quad (6)$$

In this expression  $q = e^\eta$  parameterizes the anisotropy parameter,  $\Delta = (q + q^{-1})/2$ . In order to avoid case differentiations we shall assume in the following that  $\eta = -i\gamma$ ,  $\gamma \in (0, \pi/2)$ . Then  $0 < \Delta < 1$ . We shall comment on the case  $-1 < \Delta < 0$  below and plan to treat the ‘massive case’  $\Delta > 1$  in a separate work. The integer  $s$  refers to the  $z$ -component of the pseudo spin of the excited state of the quantum transfer matrix that characterizes  $\xi_n$ . For the transversal two-point functions (4) it must be equal to the eigenvalue belonging to  $\sigma^+$  under the adjoint action of  $\sigma^z/2$ , *i.e.*  $s = 1$ .

The auxiliary functions  $a_n(\cdot|\alpha)$  are solutions of nonlinear integral equations,

$$\ln(a_n(\lambda|\alpha)) = -2(\kappa + \alpha + s)\eta - \beta e(\lambda) - \int_{\mathcal{C}_n} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + a_n(\mu|\alpha)). \quad (7)$$

As usual in integrable systems everything here is determined by one- and two-particle data. The function

$$e(\lambda) = \text{cth}(\lambda) - \text{cth}(\lambda + \eta) \quad (8)$$

is the bare energy, and the kernel

$$K(\lambda) = \text{cth}(\lambda - \eta) - \text{cth}(\lambda + \eta) \quad (9)$$

is the derivative of the bare two-particle scattering phase. The twist  $\alpha \in \mathbb{C}$  is an important regularization parameter as will become clear below. It has a physical meaning in the conformal limit [5]. In equation (7) it appears as a shift of the rescaled magnetic field  $\kappa$ . This field and the rescaled inverse temperature  $\beta$  are defined as

$$\kappa = \frac{h}{2\eta T}, \quad \beta = \frac{2J \text{sh}(\eta)}{T}. \quad (10)$$

The physical correlation functions are obtained in the limit  $\alpha \rightarrow 0$ . For this reason we may assume that  $0 < |\alpha| \ll 1$ .

Solutions of (7) are characterized by equivalence classes of contours  $\mathcal{C}_n$ , two contours being equivalent if they admit the same solution  $a_n(\cdot|\alpha)$ . The ‘canonical contour’  $\mathcal{C}_0$  associated with the dominant eigenvalue of the quantum transfer matrix consists of two straight lines parallel to the real axis and passing through  $\mp i(\gamma/2 - 0^+)$ . It is oriented in such a way that it encircles the real axis in a counterclockwise manner. The corresponding untwisted auxiliary function, the solution  $a_0 = a_0(\cdot|0)$  of (7) in the pseudo spin  $s = 0$  sector, characterizes the thermal equilibrium of the model. It parameterizes the free energy per lattice site [25], the multiple-integral representations of temperature correlation functions [12, 13] and the physical part of the temperature correlation functions in factorized form [2]. For  $s \neq 0$  there are reference contours

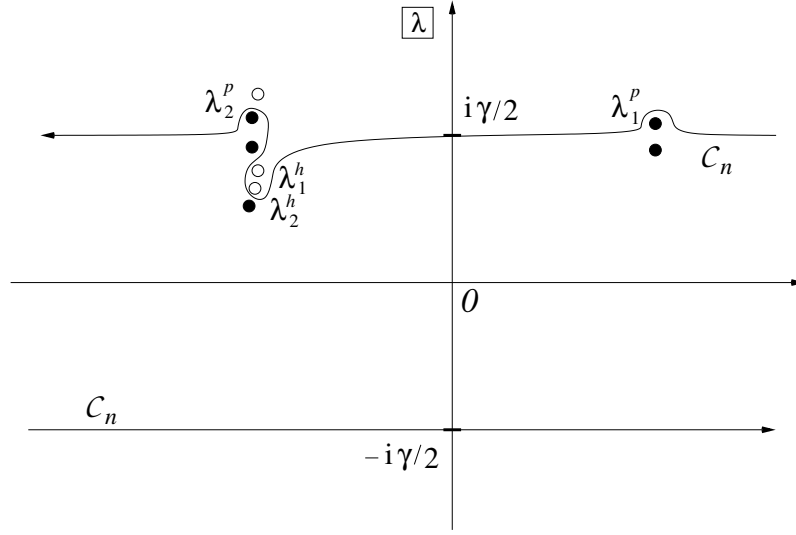


Figure 1: Sketch of a contour  $\mathcal{C}_n$  for  $0 < \Delta < 1$ . Holes  $\lambda_j^h$  inside the ‘fundamental strip’ between  $-i\gamma/2$  and  $i\gamma/2$  are excluded from the contour, and particles  $\lambda_k^p$  outside the fundamental strip are included into the contour.

$\mathcal{C}_{0,s}$  associated with the eigenvalue of largest modulus in the respective sector. The case  $s = 1$  will be described in detail below. The contours  $\mathcal{C}_n$  are deformations of the reference contours. Zeros  $\lambda_j^h$  of  $1 + \mathfrak{a}_n(\cdot|\alpha)$  inside  $\mathcal{C}_{0,s}$  but outside  $\mathcal{C}_n$  will be called holes, and zeros  $\lambda_k^p$  of  $1 + \mathfrak{a}_n(\cdot|\alpha)$  outside  $\mathcal{C}_{0,s}$  but inside  $\mathcal{C}_n$  will be called particles.

Sets of zeros and holes and equivalence classes of contours are in one-to-one correspondence. This can be seen by deforming the contours in (7) into the reference contours  $\mathcal{C}_{0,s}$  (‘straightening the contours’). Then the particles and holes appear as explicit parameters in the driving terms of the integral equation and must be determined by the subsidiary conditions  $1 + \mathfrak{a}_n(\lambda^{p,h}|\alpha) = 0$ . This formulation will be useful in section 3 on the low-temperature limit.

Expressions for the amplitudes  $A_n^{-+}$  in the form-factor expansion of the transversal correlation functions were obtained in [9],  $A_n^{-+} = \lim_{\alpha \rightarrow 0} A_n^{-+}(0|\alpha)$ , where

$$A_n^{-+}(\xi|\alpha) = \frac{\overline{G}_+^-(\xi)\overline{G}_-^+(\xi)}{(q^{1+\alpha} - q^{-1-\alpha})(q^\alpha - q^{-\alpha})} \times \exp \left\{ - \int_{\mathcal{C}_n} \frac{d\lambda}{2\pi i} \ln(\rho_n(\lambda|\alpha)) \partial_\lambda \ln \left( \frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)} \right) \right\} \times \frac{\det_{dm_+^{\alpha}, \mathcal{C}_n} \{1 - \widehat{K}_{1-\alpha}\} \det_{dm_-^{\alpha}, \mathcal{C}_n} \{1 - \widehat{K}_{1+\alpha}\}}{\det_{dm_0^{\alpha}, \mathcal{C}_n} \{1 - \widehat{K}\} \det_{dm, \mathcal{C}_n} \{1 - \widehat{K}\}}. \quad (11)$$

Here, for  $s = \pm$ ,

$$\overline{G}_s^{\pm}(\xi) = \lim_{\text{Re } \lambda \rightarrow \pm\infty} \overline{G}_s(\lambda, \xi) \quad (12)$$

and  $\bar{G}_\pm(\lambda, \xi)$  is the solution of the linear integral equation

$$\begin{aligned} \bar{G}_\pm(\lambda, \xi) = & -\text{cth}(\lambda - \xi) + q^{\alpha \mp 1} \rho_n^{\pm 1}(\xi|\alpha) \text{cth}(\lambda - \xi - \eta) \\ & + \int_{\mathbb{C}_n} dm_\pm^\alpha(\mu) \bar{G}_\pm(\mu, \xi) K_{\alpha \mp 1}(\mu - \lambda) \end{aligned} \quad (13)$$

with deformed kernel

$$K_\alpha(\lambda) = q^{-\alpha} \text{cth}(\lambda - \eta) - q^\alpha \text{cth}(\lambda + \eta). \quad (14)$$

Note that  $\bar{G}_-^+$  is analytic in  $\alpha$  and that  $\bar{G}_-^+|_{\alpha=0} = 0$  which implies that the limit  $\alpha \rightarrow 0$  exists in (11). The vanishing of  $\bar{G}_-^+$  at  $\alpha = 0$  came out rather indirectly in the derivation of (11) in [9]. We wish to point out that it can be shown more directly using the technique developed in [3].

The ‘measures’  $dm_\varepsilon^\alpha$ ,  $\varepsilon = -, 0, +$  and  $dm$  are defined by

$$dm_-^\alpha(\lambda) = \frac{d\lambda \rho_n^{-1}(\lambda|\alpha)}{2\pi i(1 + a_0(\lambda))}, \quad dm_+^\alpha(\lambda) = \frac{d\lambda \rho_n(\lambda|\alpha)}{2\pi i(1 + a_n(\lambda|\alpha))}, \quad (15a)$$

$$dm(\lambda) = \frac{d\lambda}{2\pi i(1 + a_0(\lambda))}, \quad dm_0^\alpha(\lambda) = \frac{d\lambda}{2\pi i(1 + a_n(\lambda|\alpha))}. \quad (15b)$$

The determinants in (11) are Fredholm determinants of integral operators defined by the respective kernels and measures and by the integration contours  $\mathbb{C}_n$  (see [9] for more details).

We would like to emphasize that the amplitudes are entirely described in terms of functions that appeared earlier in the description of the thermodynamic properties, the correlation length and the correlation functions of the model. These are the auxiliary functions  $a_n$  [24, 25], the eigenvalue ratios  $\rho_n$  [2, 15] and the deformed kernel  $K_\alpha$  [4]. The parameter  $s$  corresponds to the spins of the operators  $\sigma^\pm$  occurring in the transversal correlation function (see [9]).

### 3 The low-temperature limit

Equations (6) and (11) for the correlation lengths and amplitudes in the asymptotic series (4) are valid for all positive temperatures and magnetic fields and, with an appropriate adaption of the definition of the contours, also for all  $\Delta > -1$ . In the general case we do not expect any further simplification of these formulae. Their further analysis will have to rely on numerical calculations. In the low-temperature limit, for  $|\Delta| < 1$  and  $h$  below the critical field in particular, we expect universal behaviour described by conformal field theory, and further simplification should be possible.

A similar low-temperature analysis was carried out for a generating function of the density-density correlation functions of the Bose gas with delta-function interaction [28, 29] and for a generating function of the longitudinal two-point functions of the XXZ chain [9]. We shall see that the latter analysis basically carries over to the case of the transversal correlation functions considered in this work. It consists of the following steps. We first straighten the contours in the nonlinear integral equations (7) and in the expressions of the correlation lengths (6) and amplitudes (11), making the dependence

on particle and hole positions explicit. Then we perform a low-temperature analysis of the nonlinear integral equations (7) using a generalization of Sommerfeld's lemma [40]. Note that this is only possible for non-zero magnetic field as it relies on the existence of a pair of Fermi points. The low-temperature data from the analysis of the nonlinear integral equations then enter the analysis of the correlation lengths and amplitudes. It turns out that infinitely many correlation lengths diverge in the low-temperature limit and that the corresponding amplitudes all vanish as  $T$  goes to zero. Therefore, using a formula derived in [16, 20, 34], we first sum up the leading low-temperature contributions which then allows us to perform the zero-temperature limit. We shall take the opportunity to present some details and proofs that were omitted in our previous work [9].

### 3.1 Straightening the contours

We denote the number of holes by  $n_h$  and the number of particles by  $n_p$  and define the functions

$$E(\lambda) = \ln\left(\frac{\text{sh}(\lambda)}{\text{sh}(\eta + \lambda)}\right), \quad \theta(\lambda) = \ln\left(\frac{\text{sh}(\eta - \lambda)}{\text{sh}(\eta + \lambda)}\right) \quad (16)$$

which are antiderivatives of the bare energy  $e(\lambda)$  and of the kernel  $K(\lambda)$ . We further introduce the shorthand notation

$$z(\lambda) = -\frac{1}{2\pi i} \ln\left(\frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)}\right). \quad (17)$$

A straightening of the contours in (7) leads to

$$\begin{aligned} \ln(\mathfrak{a}_n(\lambda|\alpha)) &= i\pi s - 2(\kappa + \alpha)\eta - \beta e(\lambda) \\ &+ \sum_{j=1}^{n_h} \theta(\lambda - \lambda_j^h) - \sum_{j=1}^{n_p} \theta(\lambda - \lambda_j^p) - \int_{\mathcal{C}_{0,s}} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + \mathfrak{a}_n(\mu|\alpha)). \end{aligned} \quad (18)$$

Here the contour  $\mathcal{C}_{0,s}$  is chosen in such a way that

$$n_h - n_p = s. \quad (19)$$

In the free fermion limit  $\gamma \rightarrow \pi/2$  and in the low-temperature limit (see below) we shall see that  $\mathcal{C}_{0,0}$  is equal to the canonical contour  $\mathcal{C}_0$ . We shall also find an explicit description of  $\mathcal{C}_{0,1}$ . Equation (18) defines an  $(n_p + n_h)$ -parametric family of functions, depending on  $\{\lambda_j^h\}$  and  $\{\lambda_j^p\}$ . The individual functions  $\mathfrak{a}_n(\cdot|\alpha)$  are then determined by the subsidiary conditions

$$\mathfrak{a}_n(\lambda_j^h|\alpha) = \mathfrak{a}_n(\lambda_k^p|\alpha) = -1, \quad j = 1, \dots, n_h, \quad k = 1, \dots, n_p \quad (20)$$

fixing these parameters to a discrete set of values.

Straightening the contours the eigenvalue ratios take the form

$$\rho_n(\lambda|\alpha) = q^\alpha \exp\left\{ \sum_{j=1}^{n_h} E(\lambda_j^h - \lambda) - \sum_{j=1}^{n_p} E(\lambda_j^p - \lambda) - \int_{\mathcal{C}_{0,s}} d\mu e(\mu - \lambda) z(\mu) \right\} \quad (21)$$

for  $\lambda$  inside  $\mathcal{C}_{0,s}$ .

For the amplitudes we concentrate on the exponential term in (11). Upon straightening the contours it turns into

$$\begin{aligned}
A_n^{(0)}(\alpha) &= \exp \left\{ - \int_{\mathcal{C}_n} \frac{d\lambda}{2\pi i} \ln(\rho_n(\lambda|\alpha)) \partial_\lambda \ln \left( \frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)} \right) \right\} \\
&= (-1)^s q^{s\alpha} \frac{\prod_{j=1}^{n_h} \prod_{k=1}^{n_p} \text{sh}(\lambda_j^h - \lambda_k^p + \eta) \text{sh}(\lambda_j^h - \lambda_k^p - \eta)}{\left[ \prod_{j,k=1}^{n_h} \text{sh}(\lambda_j^h - \lambda_k^h - \eta) \right] \left[ \prod_{j,k=1}^{n_p} \text{sh}(\lambda_j^p - \lambda_k^p + \eta) \right]} \\
&\quad \times \frac{\left[ \prod_{\substack{j=1 \\ j \neq k}}^{n_h} \text{sh}(\lambda_j^h - \lambda_k^h) \right] \left[ \prod_{\substack{j=1 \\ j \neq k}}^{n_p} \text{sh}(\lambda_j^p - \lambda_k^p) \right]}{\left[ \prod_{j=1}^{n_h} \prod_{k=1}^{n_p} \text{sh}(\lambda_j^h - \lambda_k^p) \right]^2} \\
&\quad \times \left[ \prod_{j=1}^{n_p} \left( \partial_\lambda e^{-2\pi i z(\lambda)} \right)_{\lambda=\lambda_j^p}^{-1} \right] \left[ \prod_{j=1}^{n_h} \left( \partial_\lambda e^{-2\pi i z(\lambda)} \right)_{\lambda=\lambda_j^h}^{-1} \right] \\
&\quad \times \exp \left\{ \sum_{j=1}^{n_p} \int_{\mathcal{C}_{0,s}} d\lambda z(\lambda) (e(\lambda - \lambda_j^p) - e(\lambda_j^p - \lambda)) \right\} \\
&\quad \times \exp \left\{ \sum_{j=1}^{n_h} \int_{\mathcal{C}_{0,s}} d\lambda z(\lambda) (e(\lambda_j^h - \lambda) - e(\lambda - \lambda_j^h)) \right\} \\
&\quad \times \exp \left\{ - \int_{\mathcal{C}_{0,s}} d\lambda \int_{\mathcal{C}'_{0,s}} d\mu z(\lambda) e'(\lambda - \mu) z(\mu) \right\}. \tag{23}
\end{aligned}$$

Here  $\mathcal{C}'_{0,s}$  is a contour infinitesimally close to  $\mathcal{C}_{0,s}$  and inside  $\mathcal{C}_{0,s}$ . The Fredholm determinants and the factors  $\overline{G}_\pm$  need a different treatment which will be discussed below. In [9] we called the above contribution to the amplitude the ‘universal part’, since it is of the same form in the longitudinal and transversal case.

### 3.2 Low-temperature analysis of the nonlinear integral equations

In [9] we performed a low-temperature analysis of the nonlinear integral equations (18) that also suits for our present purposes. Here we shall reproduce this analysis and take the opportunity to add more details, to make our arguments more rigorous and to clarify the appearance and disappearance of certain phases ‘ $p$ ’.

We introduce the notations

$$\varepsilon_0(\lambda) = h - \frac{4J(1 - \Delta^2)}{\text{ch}(2\lambda) - \Delta} \tag{24}$$

and

$$u_0(\lambda) = -T \ln(\mathfrak{a}_0(\lambda + i\gamma/2)), \quad u(\lambda) = -T \ln(\mathfrak{a}_n(\lambda + i\gamma/2|\alpha)), \tag{25}$$

where we omitted the index  $n$  and the dependence on  $\alpha$  in the definition of  $u$ . Then (18)



implies that

$$u(\lambda) = \varepsilon_0(\lambda) + T \left[ 2\pi i(\alpha' - s/2) + \sum_{j=1}^{n_p} \theta(\lambda - \lambda_j^p + i\gamma/2) - \sum_{j=1}^{n_h} \theta(\lambda - \lambda_j^h + i\gamma/2) \right] \\ + T \int_{\mathcal{C}_{0,s} - i\gamma/2} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln \left( 1 + e^{-\frac{u(\mu)}{T}} \right), \quad (26)$$

where  $\alpha' = \eta\alpha/i\pi$ . A similar equation without the contribution proportional to  $T$  in the driving term holds for  $u_0$ ,

$$u_0(\lambda) = \varepsilon_0(\lambda) + T \int_{\mathcal{C}_0 - i\gamma/2} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln \left( 1 + e^{-\frac{u_0(\mu)}{T}} \right). \quad (27)$$

Since  $\theta$  is bounded on the contour  $\mathcal{C}_{0,s} - i\gamma/2$ , the terms in square brackets in (26) may be neglected compared to  $\varepsilon_0(\lambda)$  when  $T$  becomes small. Thus,  $u$  and  $u_0$  have the same zero temperature limit  $\varepsilon$ .

Intuitively the zero-temperature limit of (26) and (27) is rather clear: the integrals in (26) and (27) vanish on those parts of the contour on which  $\text{Re } \varepsilon > 0$  and are nonzero on their complement. From the behaviour of the driving term  $\varepsilon_0(\lambda)$  one may guess that this complement is an interval  $[-Q, Q]$  on the real axis. This can be stated more precisely. For  $0 < h < h_c = 4J(1 + \Delta)$  we define the dressed energy  $\varepsilon$  as the solution of the linear integral equation

$$\varepsilon(\lambda) = \varepsilon_0(\lambda) + \int_{-Q}^Q \frac{d\mu}{2\pi i} K(\lambda - \mu) \varepsilon(\mu), \quad (28)$$

where  $Q > 0$  as a function of  $h$  is uniquely determined by the condition  $\varepsilon(Q) = 0$  (for a proof of the latter statement see [8]).  $\varepsilon$  is real and even on  $\mathbb{R}$  and monotonously increasing on  $\mathbb{R}_+$ . One can further prove [8] that, for all  $\gamma \in (0, \pi/2)$ ,  $\text{Re } \varepsilon > h/4 > 0$  on  $\mathbb{R} - i\gamma + i0$  which is the lower part of the integration contour in (26), (27).

Assuming uniqueness of the solutions of (26) and (27) the above properties of  $\varepsilon$  allow us to conclude that  $\lim_{T \rightarrow 0} u(\lambda) = \lim_{T \rightarrow 0} u_0(\lambda) = \varepsilon(\lambda)$ . This follows from the following ‘generalized Sommerfeld lemma’ which also allows us to obtain the first and second order temperature corrections below.

**Lemma 1.** *Let  $T > 0$ . Let  $u, f$  be holomorphic in an open set containing a contour  $\mathcal{C}_u$ , and  $f$  bounded on  $\mathcal{C}_u$ . Let  $\ln(1 + e^{-u(\lambda)/T})$  be continuous on  $\mathcal{C}_u$  (this means we consider  $\mathcal{C}_u$  as a contour on the Riemann surface of  $\ln(1 + e^{-u(\lambda)/T})$  realized as a multi-sheeted cover of the complex plane with cuts along the curves where the argument of the logarithm is negative, say). Let  $v = \text{Re } u$ ,  $w = \text{Im } u$ . Assume that  $v$  has exactly two zeros  $Q_{\pm}$  on  $\mathcal{C}_u$  dividing  $\mathcal{C}_u$  into two parts,  $\mathcal{C}_u^-$  on which  $v$  is negative and  $\mathcal{C}_u^+$  on which  $v$  is positive. Let  $\mathcal{C}_u$  be oriented in such a way that  $Q_-$  comes before  $Q_+$  on  $\mathcal{C}_u^-$ . If there is a  $p \in \mathbb{Z}$  such that  $w(Q_{\pm}) = 2\pi pT$  then there is a choice of branches of  $\ln(1 + e^{-u(\lambda)/T})$  for which*

$$T \int_{\mathcal{C}_u} d\lambda f(\lambda) \ln \left( 1 + e^{-\frac{u(\lambda)}{T}} \right) = - \int_{Q_-}^{Q_+} d\lambda f(\lambda) (u(\lambda) - 2\pi i p T) \\ + \frac{T^2 \pi^2}{6} \left( \frac{f(Q_+)}{u'(Q_+)} - \frac{f(Q_-)}{u'(Q_-)} \right) + \mathcal{O}(T^4). \quad (29)$$

*Proof.* Let  $F$  be an antiderivative of  $f$ . Then

$$\begin{aligned} T \int_{\mathcal{C}_u^-} d\lambda f(\lambda) \ln\left(1 + e^{-\frac{u(\lambda)}{T}}\right) \\ = \int_{\mathcal{C}_u^-} d\lambda \left\{ T f(\lambda) \ln\left(1 + e^{-\frac{u(\lambda)}{T}}\right) - f(\lambda) u(\lambda) + T \partial_\lambda F(\lambda) g(\lambda) \right\}, \end{aligned} \quad (30)$$

where

$$g(\lambda) = \frac{u(\lambda)}{T} + \ln\left(1 + e^{-\frac{u(\lambda)}{T}}\right) - \ln\left(1 + e^{\frac{u(\lambda)}{T}}\right). \quad (31)$$

This function is continuous on the contour  $\mathcal{C}_u^-$  no matter how we fix the branches of the logarithm, since the first two terms on the right hand side are continuous by hypothesis and since  $\operatorname{Re}\left(1 + e^{u(\lambda)/T}\right) \geq 0$  on  $\mathcal{C}_u^-$ . On the other hand  $\exp\{g(\lambda)\} = 1 \Rightarrow \exists n \in \mathbb{Z}$  such that  $g(\lambda) = 2\pi i n$ . Thus, no matter how we chose the branches of the logarithms, there is always an  $n \in \mathbb{Z}$  such that

$$\begin{aligned} T \int_{\mathcal{C}_u^-} d\lambda f(\lambda) \ln\left(1 + e^{-\frac{u(\lambda)}{T}}\right) \\ = T \int_{\mathcal{C}_u^-} d\lambda f(\lambda) \ln\left(1 + e^{-\frac{u(\lambda)}{T}}\right) - \int_{\mathcal{C}_u^-} d\lambda f(\lambda) (u(\lambda) - 2\pi i n T). \end{aligned} \quad (32)$$

Our goal is to estimate the integral over  $\mathcal{C}_u$  for small positive  $T$ . If  $\lambda \in \mathcal{C}_u^- \setminus \{Q_-, Q_+\}$  then  $v(\lambda) < 0$  and  $e^{u(\lambda)/T} = \mathcal{O}(T^\infty)$ . Moreover, if we fix the branch of  $\ln(1 + e^{u(Q_-)/T})$  such that  $|\arg(1 + e^{u(Q_-)/T})| < \pi/2 \Rightarrow |\arg(1 + e^{u(\lambda)/T})| < \pi/2$  for all  $\lambda \in \mathcal{C}_u^-$ , and the first term on the right hand side of (32) has no  $\mathcal{O}(T)$  contribution. On any part of  $\mathcal{C}_u$  that is disconnected with  $Q_-$  a similar argument applies. Since  $v(\lambda) > 0$  on these parts, we may assume that  $|\arg(1 + e^{-u(\lambda)/T})| < \pi/2$  and that there is no  $\mathcal{O}(T)$  contribution. For the part of  $\mathcal{C}_u^+$  that is connected with  $Q_-$  we chose the branch in such a way that  $|\arg(1 + e^{-u(Q_-)/T})| < \pi/2$ . Then there is again no  $\mathcal{O}(T)$  contribution. Now if

$$u(Q_+) = u(Q_-) = 2\pi i p T, \quad (33)$$

for some  $p \in \mathbb{Z}$  then (31) with  $\lambda = Q_-$  implies that  $g(Q_-) = 2\pi i n = 2\pi i p$ . The logarithms cancel each other because of our choice of branches. Using once more (31), this time at  $Q_+$ , we conclude that

$$\ln\left(1 + e^{-\frac{u(Q_+)}{T}}\right) = \ln\left(1 + e^{\frac{u(Q_+)}{T}}\right). \quad (34)$$

Thus, at  $\lambda = Q_+$  we can continue  $\ln(1 + e^{u(\lambda)/T})$  on  $\mathcal{C}_u^-$  continuously into  $\ln(1 + e^{-u(\lambda)/T})$  on  $\mathcal{C}_u^+$ , and  $|\arg(1 + e^{-u(Q_+)/T})| < \pi/2$  which means that there is no  $\mathcal{O}(T)$  contribution on the part of  $\mathcal{C}_u^+$  connected with  $Q_+$  either.

It follows from (32) that

$$\begin{aligned} \Delta I := T \int_{\mathcal{C}_u} d\lambda f(\lambda) \ln\left(1 + e^{-\frac{u(\lambda)}{T}}\right) + \int_{Q_-}^{Q_+} d\lambda f(\lambda) (u(\lambda) - 2\pi i p T) \\ = T \int_{\mathcal{C}_u} d\lambda f(\lambda) \ln\left(1 + e^{-\frac{u(\lambda) \operatorname{sign}(v(\lambda))}{T}}\right), \end{aligned} \quad (35)$$

where the logarithm on the right hand side is continuous at  $Q_\pm$  and where the real part of the exponent on the right hand side is negative everywhere except at  $Q_\pm$ . This means

that the leading contribution to the integral for  $T \rightarrow 0$  comes from the (infinitesimally small) vicinities of these two points. In order to quantify the leading contribution we fix  $\delta > 0$  small enough. Since  $u$  and  $f$  are holomorphic we can deform the contour locally in a small vicinity of  $Q_{\pm}$  into contours  $J_{\pm}^{\delta}$  such that  $w(\lambda) = 2\pi pT$  for  $\lambda \in J_{\pm}^{\delta}$  and  $v(\lambda) = \pm\delta$  at the boundaries of  $J_{\pm}^{\delta}$ . Note that (for  $\delta$  small enough)  $v$  is monotonic on  $J_{\pm}^{\delta}$ , since it has simple zeros at  $Q_{\pm}$ . It follows that

$$\Delta I = T \int_{J_{-}^{\delta} \cup J_{+}^{\delta}} d\lambda f(\lambda) \ln \left( 1 + e^{-\frac{|v(\lambda)|}{T}} \right) + \mathcal{O}(T^{\infty}). \quad (36)$$

We parameterize  $J_{\pm}^{\delta}$  by  $x = v(\lambda) \Leftrightarrow \lambda = v^{-1}(x)$ . Then

$$\begin{aligned} T \int_{J_{-}^{\delta}} d\lambda f(\lambda) \ln \left( 1 + e^{-\frac{|v(\lambda)|}{T}} \right) &= T \int_{\delta}^{-\delta} dx \frac{f(v^{-1}(x))}{v'(v^{-1}(x))} \ln \left( 1 + e^{-\frac{|x|}{T}} \right) \\ &= -\frac{T^2 \pi^2}{6} \frac{f(Q_-)}{u'(Q_-)} + \mathcal{O}(T^4). \end{aligned} \quad (37)$$

Here we have used that  $v^{-1}(0) = Q_-$  and that  $\ln(1 + e^{-|x|})$  is even. At  $Q_+$  we can perform a similar calculation with the only difference that  $v(\lambda)$  is ascending in the direction of the contour, whence the sign will be positive.  $\square$

This lemma can be directly applied to the function  $\varepsilon(\lambda)$  defined by (28) which meets the requirements of the lemma with  $\mathcal{C}_{\varepsilon} = \mathcal{C}_0 - i\gamma/2$ ,  $Q_{\pm} = \pm Q$  and  $p = 0$ . Thus,

$$\varepsilon(\lambda) - \varepsilon_0(\lambda) - T \int_{\mathcal{C}_0 - i\gamma/2} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln \left( 1 + e^{-\frac{\varepsilon(\mu)}{T}} \right) = \mathcal{O}(T^2). \quad (38)$$

Comparing with (26), (27) we see that asymptotically for small  $T$  these equations are indeed satisfied by  $\varepsilon$ . Equivalently,  $\alpha_n(\lambda|\alpha) \sim \alpha_0(\lambda) \sim e^{-\varepsilon(\lambda - i\gamma/2)/T}$ .

The function  $u_0$  characterizing the dominant eigenvalue is approximated by  $\varepsilon$  with an error of second order in  $T$ . Hence, up to terms of the order of  $T^2$ , the zeros of  $1 + \alpha_0$  are determined by

$$\varepsilon(\lambda) = -(2n-1)\pi iT, \quad n \in \mathbb{Z}. \quad (39)$$

Recall that those zeros that are located below the real axis are the Bethe roots (shifted by  $-i\gamma/2$ ) of the dominant state of the quantum transfer matrix in the Trotter limit (see e.g. [12]).

The solutions of equation (39) for a specific choice of parameters are shown in Figure 2. On the closed curve encircling the origin and intersecting the real axis at  $\pm 2$  the function  $\varepsilon$  is purely imaginary. This closed curve  $\Gamma$  is intersected by the curves  $\Gamma_n$  of constant imaginary part  $\text{Im}(\varepsilon(\lambda)) = -(2n-1)\pi T$ . The intersection points are the solutions of (39). The function  $\varepsilon$  has two zeros at  $\pm Q = \pm 2$  and two poles at  $\pm i\gamma/2 = \pm 0.580i$  on  $\Gamma$ . On each of the two arcs, running from  $-i\gamma/2$  through  $Q$  to  $i\gamma/2$  and back from  $i\gamma/2$  through  $-Q$  to  $-i\gamma/2$ , the values of  $\varepsilon$  increase from  $-i\infty$  to  $+i\infty$ . Thus, there are infinitely many solutions of (39) on  $\Gamma$  clustering at  $\pm i\gamma/2$ . The figure shows only the solutions closest to the real axis.

From the picture we can also understand the qualitative behaviour of the function  $\alpha_0$ :  $|\alpha_0| = 1$  on  $\Gamma$ ,  $|\alpha_0(\lambda)| > 1$  for  $\lambda$  inside  $\Gamma$  and  $|\alpha_0(\lambda)| < 1$  for  $\lambda$  outside  $\Gamma$ . Moreover,  $\alpha_0$  is real negative on  $\Gamma_n$  with  $\alpha_0(\lambda) < -1$  for  $\lambda$  inside  $\Gamma$  and  $-1 < \alpha_0(\lambda) < 0$  for  $\lambda$  outside  $\Gamma$ .

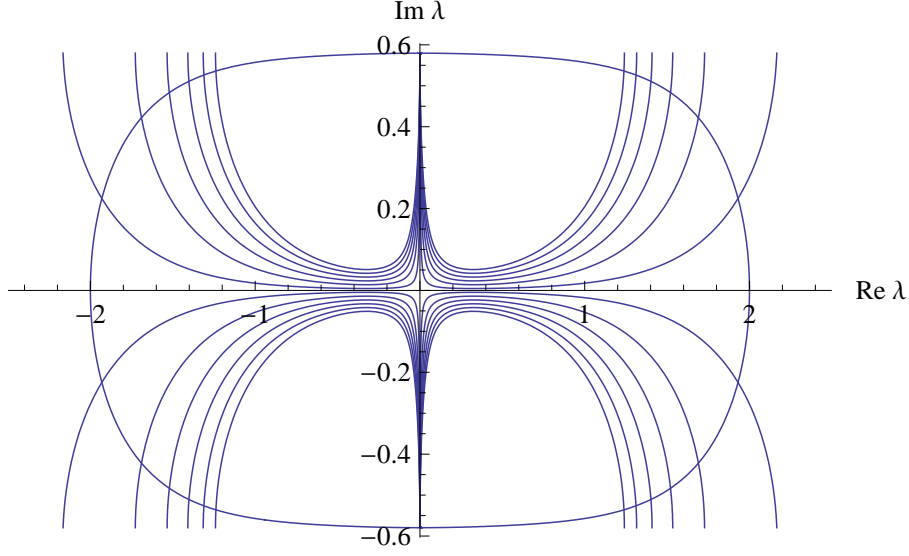


Figure 2: Solutions of the equation  $\varepsilon(\lambda) = -(2n-1)\pi iT$  in the complex plane depicted by the intersections of the curve  $\text{Re } \varepsilon(\lambda) = 0$  (closed curve encircling the origin) with the curves  $\text{Im } \varepsilon(\lambda) = -(2n-1)\pi T$  (open curves).  $J = 1$ ,  $T = 0.01$ ,  $\Delta = 0.4$ ,  $h = 0.051$  and  $n = -5, -4, \dots, 6$  in this example. The solutions form an infinite sequence of points on the line  $\text{Re } \varepsilon(\lambda) = 0$  with two limit points at  $\pm i\gamma/2$  (here  $\gamma/2 = 0.580$ ). Shown are only the points farthest away from these limit points.

This means that the  $\Gamma_n$  are the canonical cuts needed to construct the Riemann surface of the function  $\ln(\alpha_0)$ . It further follows that the function  $1 + \alpha_0$  is real and negative with range between  $-\infty$  and 0 on those parts of the contours  $\Gamma_n$  that are located inside  $\Gamma$ . These are therefore the cuts for the Riemann surface of  $\ln(1 + \alpha_0)$ .

Equation (39) resembles a momentum quantization condition for free Fermions with momentum replaced by energy  $-\varepsilon/T$ . The appearance of this terms seems rather natural in view of the fact that ‘space and time direction’ are interchanged in the six-vertex model representing the partition function of the XXZ chain within the quantum transfer matrix formalism.

As against  $u_0$  the functions  $u$  are generally approximated by  $\varepsilon$  only in the strict limit  $T \rightarrow 0$ , when  $(2n-1)\pi iT$  becomes a continuous variable. In this limit the possible particle and hole positions (shifted downward by  $i\gamma/2$ ) densely fill the curve  $\text{Re } \varepsilon(\lambda) = 0$ . In order to obtain discrete values to the order  $T$ , one has to take into account the  $\mathcal{O}(T)$  contribution on the right hand side of equation (26).

Assuming that for fixed particle and hole parameters the functions  $u$  admit the low-temperature asymptotic expansion

$$u(\lambda) = \varepsilon(\lambda) + Tu_1(\lambda) + T^2u_2(\lambda) + \mathcal{O}(T^3) \quad (40)$$

and that there are  $p \in \mathbb{Z}$  and  $Q_{\pm} = \pm Q + TQ_{\pm}^{(1)} + \mathcal{O}(T^2)$  such that  $u(Q_{\pm}) = 2\pi ipT$  we conclude that

$$Q_{\pm} = \pm Q \mp \frac{\bar{u}_1(\pm Q)}{\varepsilon'(Q)}T + \mathcal{O}(T^2), \quad (41)$$

where  $\bar{u}_1(\lambda) = u_1(\lambda) - 2\pi i p$  by definition. Since  $u$  is close to  $\varepsilon$  as long as  $T$  is small enough we may apply Lemma 1 to  $u$ . Using a contour  $\mathcal{C}_u$  which is a deformation of  $\mathcal{C}_0 - i\gamma/2$  such that it passes through  $Q_\pm$  and introducing the notation  $\bar{u} = u - 2\pi i p T$  the lemma implies that

$$\bar{u}(\lambda) = \varepsilon_0(\lambda) + T r_1(\lambda) + T^2 r_2(\lambda) + \int_{-Q}^Q \frac{d\mu}{2\pi i} K(\lambda - \mu) \bar{u}(\mu) + \mathcal{O}(T^3), \quad (42)$$

where

$$r_1(\lambda) = 2\pi i (\alpha' - s/2 - p) + \sum_{j=1}^{n_p} \theta(\lambda - x_j^p) - \sum_{j=1}^{n_h} \theta(\lambda - x_j^h), \quad (43)$$

$$r_2(\lambda) = \frac{i\pi}{4\varepsilon'(\mathcal{Q})} \left[ K(\lambda - \mathcal{Q}) \left( \frac{1}{3} + \frac{\bar{u}_1^2(\mathcal{Q})}{\pi^2} \right) + K(\lambda + \mathcal{Q}) \left( \frac{1}{3} + \frac{\bar{u}_1^2(-\mathcal{Q})}{\pi^2} \right) \right] \quad (44)$$

with  $x_j^{p,h} = \lambda_j^{p,h} - i\gamma/2$ .

Neglecting the  $\mathcal{O}(T^3)$  terms in (42) we obtain a linear integral equation for  $\bar{u}$ . Due to its linearity we can express its solution in terms of standard functions known from the description of the ground state properties of the XXZ chain. We obtain

$$\bar{u}_1(\lambda) = 2\pi i \left[ (\alpha' - s/2 - p) Z(\lambda) + \sum_{j=1}^{n_h} \phi(\lambda, x_j^h) - \sum_{j=1}^{n_p} \phi(\lambda, x_j^p) \right], \quad (45a)$$

$$u_2(\lambda) = \frac{i\pi}{4\varepsilon'(\mathcal{Q})} \left[ R(\lambda, \mathcal{Q}) \left( \frac{1}{3} + \frac{\bar{u}_1^2(\mathcal{Q})}{\pi^2} \right) + R(\lambda, -\mathcal{Q}) \left( \frac{1}{3} + \frac{\bar{u}_1^2(-\mathcal{Q})}{\pi^2} \right) \right]. \quad (45b)$$

The functions appearing here are the dressed charge function  $Z$ , the dressed phase  $\phi$  and the resolvent  $R$ , satisfying the linear integral equations

$$Z(\lambda) = 1 + \int_{-Q}^Q \frac{d\mu}{2\pi i} K(\lambda - \mu) Z(\mu), \quad (46a)$$

$$\phi(\lambda, \nu) = -\frac{\theta(\lambda - \nu)}{2\pi i} + \int_{-Q}^Q \frac{d\mu}{2\pi i} K(\lambda - \mu) \phi(\mu, \nu), \quad (46b)$$

$$R(\lambda, \nu) = K(\lambda - \nu) + \int_{-Q}^Q \frac{d\mu}{2\pi i} K(\lambda - \mu) R(\mu, \nu). \quad (46c)$$

For later convenience we also introduce the root density  $\rho$  as the solution of

$$\rho(\lambda) = -\frac{e(\lambda + i\gamma/2)}{2\pi i} + \int_{-Q}^Q \frac{d\mu}{2\pi i} K(\lambda - \mu) \rho(\mu). \quad (47)$$

Knowing  $u_1$  we know the subsidiary conditions (20) for  $a_n(\cdot|\alpha)$  to linear order in  $T$ ,

$$\varepsilon(x_j^{p,h}) + T u_1(x_j^{p,h}) = -(2n_j^{p,h} - 1) i\pi T. \quad (48)$$

We insert the explicit expression (45a) for  $\bar{u}_1$  into this equation and obtain a set of coupled nonlinear algebraic equations for the particle and hole parameters,

$$\begin{aligned} \frac{\varepsilon(x_j^{p,h})}{2\pi i T} = & -n_j^{p,h} + 1/2 - p \\ & - (\alpha' - s/2 - p) Z(x_j^{p,h}) + \sum_{k=1}^{n_p} \phi(x_j^{p,h}, x_k^p) - \sum_{k=1}^{n_h} \phi(x_j^{p,h}, x_k^h). \end{aligned} \quad (49)$$

These equations may be interpreted as a dressed version of the logarithmic form of the Bethe ansatz equations (with dressed momentum replaced by dressed energy and  $1/L$  replaced by  $T$ ). The bare two-particle scattering phases are replaced by the dressed phases and the dressed charge appears in addition. Equations (49) have to be solved numerically for the particle and hole parameters  $x_j^p$  and  $x_j^h$ .

Simplifications occur in two cases. In the XX or free fermion case,  $\gamma = \pi/2$ ,  $\Delta = 0$ , the dressed phases vanish, the dressed charge equals one, and the dressed energy turns into the bare energy  $\varepsilon_0$ . Thus,

$$\varepsilon_0(x_j^{p,h}) = -(2n_j^{p,h} - 1 - s + \alpha')\pi iT. \quad (50)$$

The same set of decoupled equations is obtained from the non-linear integral equations (26) if one sets  $\gamma = \pi/2$ , meaning that in the XX case it is valid for any  $T$ . In this case all solutions fall into two classes depending on whether  $s/2$  is integer or half-odd integer. Comparing (50) and (39) we observe that the description of the dominant state of the quantum transfer matrix in terms of the zeros of the auxiliary function  $1 + \alpha_0$  is very close to the free fermion paradigm. This provides a useful ‘almost free fermion picture’ for the understanding of the excited states of the quantum transfer matrix at low temperatures.

Further simplifications for generic  $\gamma \in (0, \pi/2)$  occur if we restrict ourselves to excitations close to the Fermi surface consisting of the two points  $\pm Q$ . Our low-temperature analysis of the correlation functions is based on the hypothesis that these excitations contribute predominantly to the large-distance asymptotics. More precisely, we shall restrict ourselves in the following to particle and hole parameters which collapse to the Fermi points  $\pm Q$  as  $T$  goes to zero,

$$x_j^{p,h} = \pm Q + \mathcal{O}(T). \quad (51)$$

We denote their numbers by  $n_p^\pm$  and  $n_h^\pm$ , respectively, and define the particle-hole disbalance at the left Fermi point  $-Q$  by

$$\ell = n_h^- - n_p^-. \quad (52)$$

Inserting the lowest order approximation  $x_j^{p,h} = \pm Q$  into (45a) we obtain the leading low-temperature approximation to  $\bar{u}_1(\lambda)$  which we denote  $\bar{u}_1^{(\ell)}(\lambda)$ . We shall write it as

$$\bar{u}_1^{(\ell)}(\lambda) = 2\pi i (w(\lambda) + \alpha' - p - s/2), \quad (53)$$

where

$$w(\lambda) = (\alpha' - p - \ell)(Z(\lambda) - 1) + \frac{s}{2}(\phi(\lambda, Q) + \phi(\lambda, -Q)). \quad (54)$$

The function  $\bar{u}_1^{(\ell)}$  determines the points  $Q_\pm$  in (41) to linear order in  $T$ ,

$$Q_\pm = \pm Q \mp \frac{\bar{u}_1^{(\ell)}(\pm Q)}{\varepsilon'(Q)}T + \mathcal{O}(T^2). \quad (55)$$

Using (53) in (48), the equations for the particle and hole parameters decouple. We obtain

$$\frac{\varepsilon(x_j^{p,h})}{2\pi iT} = -n_j^{p,h} + \frac{1+s}{2} - \alpha' - w(x_j^{p,h}). \quad (56)$$

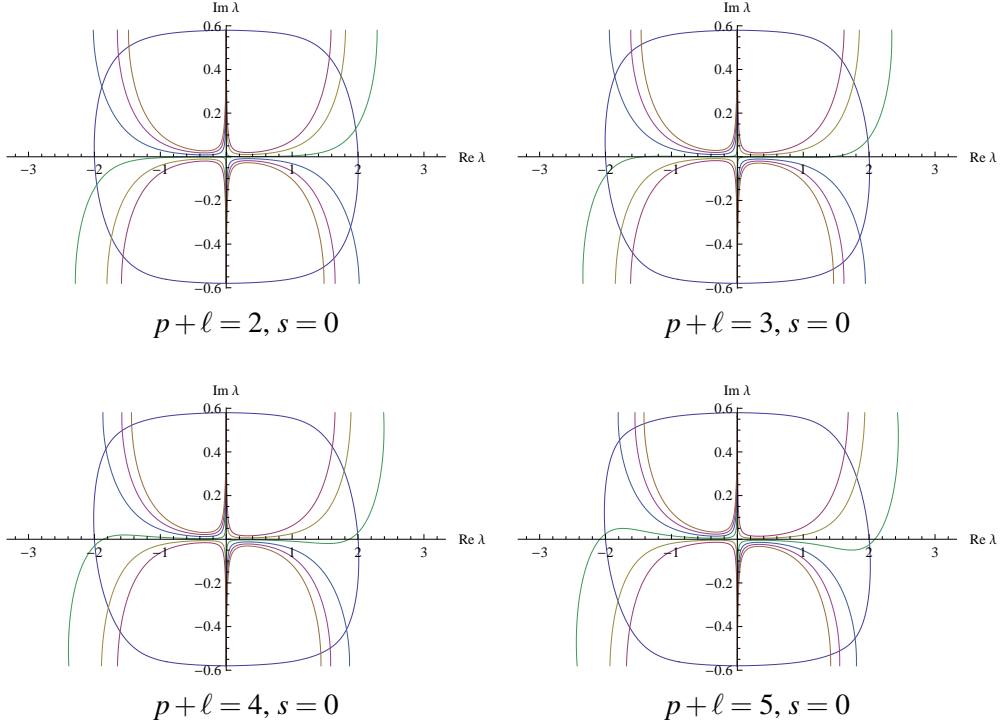


Figure 3: Graphical solutions of equation (56) in the complex plane for various values of  $p + \ell$  and  $s = 0$ . Parameter values are the same as in Figure 2:  $J = 1$ ,  $T = 0.01$ ,  $\Delta = 0.4$ ,  $h = 0.051$ . Shown are the solutions closest to the real axis.

Comparison with (50) shows that the function  $w$  determines the deviation from free-Fermion like behaviour to linear order in  $T$ . As we further see from (56), to this order, the ‘excitations above the Fermi surface’ (51) fall into classes parameterized by two sets of integers,  $p + \ell$  and  $s$ . In our case  $s = 0$  or  $s = 1$  for the longitudinal and transversal two-point functions. For fixed  $s$  all excitations above the Fermi surface are obtained by letting  $p + \ell$  run through all integers and by calculating the associated particle-hole patterns satisfying (19) and (52) from (56). We shall see below that the amplitudes that remain after summing over all particle-hole patterns for fixed  $\ell$  depend on  $p$  and  $\ell$  indeed only through their sum  $p + \ell$ .

So far we did not discuss the meaning of  $p$ . It entered our calculation when we used Lemma 1 to determine the low-temperature approximation to  $\ln a_n(\cdot|\alpha)$ . As we see from the lemma  $-2\pi p$  is the phase of  $a_n(Q_\pm|\alpha)$ . Thus, it is tightly connected with the choice of the reference contour. In principle, we would like to choose the real axis (corresponding to the upper part of the canonical contour) as a reference contour. Then  $p$  would be determined by the condition that both,  $Q_-$  and  $Q_+$  are located at the zeros of  $1 - a_n(\cdot|\alpha)$  that are closest to the real axis (*i.e.* to  $\pm Q$ ). We shall see below that such a choice of reference contour is possible for  $s = 0$ . In the general case the reference contour can pass only either through  $-Q$  or through  $Q$ . Below we shall choose  $-Q$ . Then  $Q_\pm$  are determined by (55).

In order to get an intuitive understanding of  $p$  let us consider an example. In Figure 3 we have depicted the solutions of (56) by plotting the real and imaginary parts

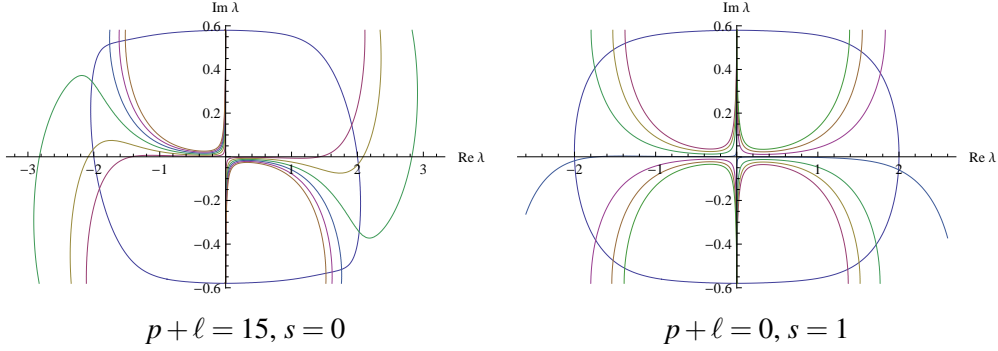


Figure 4: Graphical solutions of equation (56) in the complex plane for various values of  $p + \ell$  and  $s$ . Parameter values in the right panel  $J = 1, T = 0.0125, \Delta = 0.4, h = 0.051$  and  $\alpha = 0.3$ . In the left panel the temperature is reduced to  $T = 0.005$  and  $\alpha = 0$ .

of the difference between left and right hand side of the equation. For the same set of parameters as in Figure 2 we have set  $s = 0$  and have increased the value of  $p + \ell$  in unit steps from 2 to 5. For  $p + \ell = 0$  we would obtain again Figure 2. Possible positions of particle and hole parameters  $x_j^p, x_j^h$  are the intersection points of the open contours with the closed contours that are located above and below the real line, respectively. In this case the real axis (corresponding to the upper part of the contour  $\mathcal{C}_0 - i\gamma/2$  and run through in negative direction) is a possible reference contour, since left and right intersection points of the real axis with the curve  $\text{Re } u = 0$  are located on the same sheet of the Riemann surface of  $\ln a_n(\cdot|\alpha)$ , whose branch cuts are the open contours in the figure. As soon as  $p + \ell$  is as large as 5 in our example, our reference contour intersects one of the branch cuts of  $\ln a_n(\cdot|\alpha)$  before it intersects the closed contour corresponding to the real part of equation (56). Then the phase of  $a_n(Q_{\pm}|\alpha)$  in units of  $-2\pi$ , which is the value of  $p$ , is incremented by one to  $p = 1$ . Hence, in the first three pictures we have  $p = 0$  and  $\ell = 2, 3, 4$ , but in the last one  $p = 1$  and  $\ell = 4$ . The left panel in Figure 4 shows an example where  $p = 2$ .

For non-zero  $s$  the pictures become more asymmetric. An example is shown in the right panel of Figure 4, where  $p + \ell = 0$  and  $s = 1$ . A contour along the real line, entering the picture from the right and leaving to the left, now crosses one of the lines of constant imaginary part. Let  $Q_{\pm}$  be the zeros of  $1 - a_n(\cdot|\alpha)$  closest to the real axis. Then it follows, due to the crossing of the line, that  $\ln a_n(Q_+|\alpha) - \ln a_n(Q_-|\alpha) = -2\pi i$ . Hence, the contour does not fit the requirements of Lemma 1. A contour  $\mathcal{C}_{0,1}$  as close as possible to the real axis and meeting the requirements of the lemma must pass one of the two intersection points with negative imaginary part closest to the real axis from above and the other one from below.

From equation (56) we obtain the deviation of the particle and hole parameters  $x_j^{p,h}$  from the Fermi points to linear order in  $T$ ,

$$x_j^{p,h} = \pm Q - \left( n_j^{p,h} - \frac{1+s}{2} + \alpha' + w(\pm Q) \right) \frac{2\pi i T}{\varepsilon'(Q)}. \quad (57)$$

In this approximation the particle and hole parameters are located on lines perpendicular to the real axis and intersecting it at  $\pm Q$ .

We now distinguish particle and hole parameters pertaining the the right and left



Fermi edges  $\pm Q$ , writing  $x_j^{p\pm}$  and  $x_j^{h\pm}$ , respectively. The corresponding quantum numbers are denoted  $n_j^{p\pm}$  and  $n_j^{h\pm}$ . We reparameterize these integers by positive integers  $p_j^\pm, h_j^\pm$ , setting

$$n_j^{p\pm} = -p_j^\pm - p + 1, \quad n_j^{h\pm} = h_j^\pm - p. \quad (58)$$

In order to obtain a consistent interpretation of the positive numbers  $p_j^\pm$  and  $h_j^\pm$  as ‘particle and hole quantum numbers’ we have to fix  $p$  in such a way that e.g.  $\text{Im} x_j^{p-} > 0 > \text{Im} x_j^{h-}$  for  $p_j^- = h_j^- = 1$ . Using (53) and (57), this is equivalent to demanding that

$$|\text{Im} \bar{u}_1^{(\ell)}(-Q)| < \pi. \quad (59)$$

Inserting now (55) and (58) into (57) the particle and hole parameters become parameterized as

$$x_j^{p\pm} = Q_\pm + \left(p_j^\pm - \frac{1}{2}\right) \frac{2\pi i T}{\varepsilon'(Q)} + \mathcal{O}(T^2), \quad (60a)$$

$$x_j^{h\pm} = Q_\pm - \left(h_j^\pm - \frac{1}{2}\right) \frac{2\pi i T}{\varepsilon'(Q)} + \mathcal{O}(T^2) \quad (60b)$$

where  $p_j^\pm, h_j^\pm \in \mathbb{Z}_+$ . The inequality (59) guarantees that  $Q_-$  is as close to the real axis as possible. If  $s = 0$  we have  $\bar{u}_1^{(\ell)}(Q) = \bar{u}_1^{(\ell)}(-Q)$  and (59) automatically holds at the right Fermi edge as well. This is no longer true for  $s \neq 0$ . In that case, using the well known identities [27]

$$\mathcal{Z} = 1 + \phi(Q, Q) - \phi(Q, -Q), \quad \frac{1}{\mathcal{Z}} = 1 + \phi(Q, Q) + \phi(Q, -Q), \quad (61)$$

we can only conclude that

$$0 < \text{Im} \bar{u}_1^{(\ell)}(Q) - \text{Im} \bar{u}_1^{(\ell)}(-Q) = 2s\pi(1/\mathcal{Z} - 1) < s\pi. \quad (62)$$

The inequalities hold, since  $1/\sqrt{2} < \mathcal{Z} < 1$  as long as  $\gamma \in (0, \pi/2)$  (see e.g. [8]). Thus, for  $s = 1$  it may happen that  $\pi < \text{Im} \bar{u}_1^{(\ell)}(Q) < 2\pi$ . For  $p_j^+ = 1$  the latter implies that  $\text{Im} x_j^{p+} = (\pi - \text{Im} \bar{u}_1^{(\ell)}(Q))T/\varepsilon'(Q) < 0$ , *i.e.* according to our definition the lowest particle excitation at the right Fermi edge may correspond to an  $x_j^{p+}$  below the real axis. Of course, this is in accordance with the example considered above, illustrated in the right panel of Figure 4, and with our general picture which includes that we have to choose the contour  $\mathcal{C}_{0,1}$  carefully.

Before closing this section let us stress again that  $p$  is an auxiliary parameter associated with the choice of the integration contour, in which we have a certain freedom and which does not influence the final result of our calculation. The true parameter entering the classification of the elementary excitations is the sum  $p + \ell$ , since only this parameter enters the definition of the function  $w$  which determines the particle and hole parameters through (56).

For the low-temperature analysis of the eigenvalue ratios in the next subsection we shall need  $\bar{u}_1$  up to the first order in  $T$ . We insert (53), (60) into (45a) and use

$\partial_v \phi(\lambda, v) = R(\lambda, v)/2\pi i$  to obtain

$$\begin{aligned} \bar{u}_1(\lambda) = \bar{u}_1^{(\ell)}(\lambda) + \frac{2\pi i T}{\varepsilon'(Q)} \left\{ R(\lambda, Q) \left[ \frac{(\ell-s)\bar{u}_1^{(\ell)}(Q)}{2\pi i} - \sum_{j=1}^{n_h^+} \left( h_j^+ - \frac{1}{2} \right) - \sum_{j=1}^{n_p^+} \left( p_j^+ - \frac{1}{2} \right) \right] \right. \\ \left. + R(\lambda, -Q) \left[ \frac{\ell \bar{u}_1^{(\ell)}(-Q)}{2\pi i} - \sum_{j=1}^{n_h^-} \left( h_j^- - \frac{1}{2} \right) - \sum_{j=1}^{n_p^-} \left( p_j^- - \frac{1}{2} \right) \right] \right\} + \mathcal{O}(T^2). \quad (63) \end{aligned}$$

Equations (45b) and (63) together with the corresponding linear integral equations then determine  $\ln(a_n(\lambda|\alpha))$  up to the order  $T$ .

### 3.3 Correlation lengths and universal part of the amplitudes

Using the results of the previous subsection it is not difficult to calculate the leading order low-temperature contribution to the eigenvalue ratios for the transversal correlation functions ( $s = 1$ ). Setting  $\alpha'' = \alpha - p - \ell$  we obtain

$$\begin{aligned} \rho_n(0|\alpha) = q^\alpha \exp \left\{ i\pi - 2i\alpha'' k_F \right. \\ \left. - \frac{2\pi T}{v_0} \left[ \alpha''^2 \mathcal{Z}^2 + \frac{1}{4\mathcal{Z}^2} - \ell^2 + \ell - 1 + \sum_{j=1}^{n_h} h_j + \sum_{j=1}^{n_p} (p_j - 1) \right] \right\} + \mathcal{O}(T^2). \quad (64) \end{aligned}$$

Here we have introduced the Fermi momentum  $k_F$  and the sound velocity  $v_0$  as

$$k_F = 2\pi \int_0^Q d\lambda \rho(\lambda), \quad v_0 = \frac{\varepsilon'(Q)}{2\pi \rho(Q)}. \quad (65)$$

Starting from the expression (22) for the amplitudes and using the insight from the previous subsection we can also calculate the leading low-temperature asymptotics of the universal part of the amplitudes. After slightly tedious calculations we obtain

$$A_n^{(0)}(\alpha) = -iq^\alpha e^{E(-2Q)} A_-(\alpha) A_+(\alpha), \quad (66)$$

where

$$\begin{aligned} A_\pm(\alpha) = \exp \left\{ C_\pm[w] + \frac{1}{4} \theta(2Q) \left[ 2\alpha''^2 \mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2} + 1 - \alpha'' \left( \mathcal{Z} - \frac{1}{\mathcal{Z}} \right) \right] \right\} \\ \times \left( \frac{2\pi T e^{-E(2Q)}}{\varepsilon'(Q) \text{sh}(\eta)} \right)^{\alpha''^2 \mathcal{Z}^2 + \frac{1}{4\mathcal{Z}^2}} \frac{G(3/2 \pm (w(\pm Q) + \alpha''))}{G(1/2 \mp (w(\pm Q) + \alpha''))} \\ \times G^2 \left( 1 \mp \frac{\bar{u}_1^{(\ell)}(\pm Q)}{2\pi i} \right) \left( \frac{1}{\pi} \sin \left( \frac{\bar{u}_1^{(\ell)}(\pm Q)}{2i} \right) \right)^{2n_h^\pm} \mathcal{R}_{n_h^\pm, n_p^\pm} \left( \{h_j^\pm\}, \{p_j^\pm\} \middle| \pm \frac{\bar{u}_1^{(\ell)}(\pm Q)}{2\pi i} \right). \quad (67) \end{aligned}$$

In this expression  $G$  is the Barnes  $G$ -function and  $C_\pm$  are the functionals

$$\begin{aligned} C_\pm[v] = \frac{1}{4} \int_{-Q}^Q d\lambda \int_{-Q}^Q d\mu (v'(\lambda)v(\mu) - v(\lambda)v'(\mu)) e(\lambda - \mu) \\ \pm (v(\pm Q) \pm 1 + \alpha'') \int_{-Q}^Q d\lambda (v(\lambda) - v(\pm Q)) e(\lambda \mp Q). \quad (68) \end{aligned}$$

The functions  $\mathcal{R}$  comprise the dependence on the particle-hole quantum numbers,

$$\mathcal{R}_{n_1, n_2}(\{h_j\}, \{p_j\} | v) = \frac{\prod_{1 \leq j < k \leq n_1} (h_j - h_k)^2 \prod_{1 \leq j < k \leq n_2} (p_j - p_k)^2}{\prod_{j=1}^{n_1} \prod_{k=1}^{n_2} (h_j + p_k - 1)^2} \times \left[ \prod_{j=1}^{n_1} \frac{\Gamma^2(h_j + v)}{\Gamma^2(h_j)} \right] \left[ \prod_{j=1}^{n_2} \frac{\Gamma^2(p_j - v)}{\Gamma^2(p_j)} \right]. \quad (69)$$

The hardest part of the calculation leading to (67) is the evaluation of the singular integrals in the exponent on the right hand side of (22). It can be achieved by means of the following lemmas.

**Lemma 2.** *Let  $u$  and  $\mathcal{C}_u$  be subject to the same assumptions as in Lemma 1. Let  $\lambda_+$  be located above  $\mathcal{C}_u$  and  $\lambda_-$  below. Then the Cauchy-type integral*

$$I_u(\lambda_{\pm}) = \int_{\mathcal{C}_u} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \ln \left( 1 + e^{-\frac{u(\lambda)}{T}} \right) \quad (70)$$

*admits a low-temperature expansion whose form depends on the distance of  $\lambda_{\pm}$  from the zeros  $Q_{\pm}$  of the real part of  $u$ .*

*If  $\lambda_{\pm}$  are uniformly away from  $Q_{\pm}$ , then Lemma 1 applies, and*

$$I_u(\lambda_{\pm}) = - \int_{Q_-}^{Q_+} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \frac{\bar{u}(\lambda)}{T} + \mathcal{O}(T). \quad (71)$$

*For  $\delta > 0$  define  $V_{\pm} = \{z \in \mathbb{C} \mid |u(z)| < \delta/2, z \text{ close to } Q_{\pm}\}$ . If  $\lambda_{\pm} \in V_+$ , then there is a  $\delta > 0$  and independent of  $\lambda_{\pm}$  such that*

$$I_u(\lambda_{\pm}) = - \int_{Q_-}^{Q_+} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \frac{\bar{u}(\lambda) - \bar{u}(\lambda_{\pm})}{T} \mp 2\pi i \ln \left\{ \Gamma \left( \frac{1}{2} \pm \frac{\bar{u}(\lambda_{\pm})}{2\pi i T} \right) \right\} \\ \pm \pi i \ln(2\pi) + \frac{\bar{u}(\lambda_{\pm})}{T} \left\{ \ln \left( \frac{\bar{u}(\lambda_{\pm})}{\pm 2\pi i T} \right) - 1 - \ln \left( \frac{\operatorname{sh}(Q_+ - \lambda_{\pm})}{\operatorname{sh}(Q_- - \lambda_{\pm})} \right) \right\} + \mathcal{O}(T). \quad (72)$$

*If  $\lambda_{\pm} \in V_-$ , then*

$$I_u(\lambda_{\pm}) = - \int_{Q_-}^{Q_+} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \frac{\bar{u}(\lambda) - \bar{u}(\lambda_{\pm})}{T} \mp 2\pi i \ln \left\{ \Gamma \left( \frac{1}{2} \mp \frac{\bar{u}(\lambda_{\pm})}{2\pi i T} \right) \right\} \\ \pm \pi i \ln(2\pi) - \frac{\bar{u}(\lambda_{\pm})}{T} \left\{ \ln \left( \frac{\bar{u}(\lambda_{\pm})}{\mp 2\pi i T} \right) - 1 + \ln \left( \frac{\operatorname{sh}(Q_+ - \lambda_{\pm})}{\operatorname{sh}(Q_- - \lambda_{\pm})} \right) \right\} + \mathcal{O}(T). \quad (73)$$

**Lemma 3.**

$$- \int_{\mathcal{C}_{0,s}} d\lambda \int_{\mathcal{C}'_{0,s}} d\mu z(\lambda) \operatorname{cth}'(\lambda - \mu) z(\mu) = \\ + \ln \left\{ G \left( 1 + \frac{\bar{u}_1^{(\ell)}(Q)}{2\pi i} \right) G \left( 1 - \frac{\bar{u}_1^{(\ell)}(Q)}{2\pi i} \right) G \left( 1 + \frac{\bar{u}_1^{(\ell)}(-Q)}{2\pi i} \right) G \left( 1 - \frac{\bar{u}_1^{(\ell)}(-Q)}{2\pi i} \right) \right\} \\ + C_1 \left[ \frac{\bar{u}_1^{(\ell)}}{2\pi i} \right] - \left( \left( \frac{\bar{u}_1^{(\ell)}(Q)}{2\pi i} \right)^2 + \left( \frac{\bar{u}_1^{(\ell)}(-Q)}{2\pi i} \right)^2 \right) \ln \left( \frac{\varepsilon'(Q) \operatorname{sh}(2Q)}{2\pi T} \right) + o(1), \quad (74)$$

where  $o(1)$  denotes terms that go to zero as  $T \rightarrow 0^+$ , and

$$C_1[v] = \frac{1}{2} \int_{-Q}^Q d\lambda \int_{-Q}^Q d\mu \frac{v'(\lambda)v(\mu) - v(\lambda)v'(\mu)}{\text{th}(\lambda - \mu)} \\ + v(Q) \int_{-Q}^Q d\lambda \frac{v(\lambda) - v(Q)}{\text{th}(\lambda - Q)} - v(-Q) \int_{-Q}^Q d\lambda \frac{v(\lambda) - v(-Q)}{\text{th}(\lambda + Q)}. \quad (75)$$

Proofs of Lemma 2 and Lemma 3 are provided in Appendix A.

### 3.4 Determinant part and factorized part of the amplitudes

For the calculation of the determinant part and of the ‘factorized part’, by which we mean the product of functions  $\bar{G}_\pm$  in (11), we can closely follow Appendix D of [9]. An important property of these contributions to the amplitudes is that they do not depend on the particle and hole quantum numbers. They are functionals of  $w$  and depend only on  $p + \ell$ . Anticipating this fact we shall write

$$\mathcal{D}(p + \ell) = \lim_{T \rightarrow 0} \frac{\det_{dm_+^\alpha, \mathcal{C}_n} \{1 - \hat{K}_{1-\alpha}\} \det_{dm_-^\alpha, \mathcal{C}_n} \{1 - \hat{K}_{1+\alpha}\}}{\det_{dm_0^\alpha, \mathcal{C}_n} \{1 - \hat{K}\} \det_{dm, \mathcal{C}_n} \{1 - \hat{K}\}}. \quad (76)$$

The measures of the determinants in the denominator do not contain  $\rho_n(\cdot|\alpha)$ . Their zero temperature limit is readily understood by recalling that the weight functions  $1/(1 + \alpha_0^{-1})$  and  $1/(1 + \alpha_n^{-1}(\cdot|\alpha))$  turn into the characteristic functions of the sub-contour  $i\gamma/2 + [-Q, Q]$  on  $\mathcal{C}_0$ . Thus,

$$\lim_{T \rightarrow 0} \det_{dm_0^\alpha, \mathcal{C}_n} \{1 - \hat{K}\} = \lim_{T \rightarrow 0} \det_{dm, \mathcal{C}_n} \{1 - \hat{K}\} = \det_{d\lambda/2\pi i, [-Q, Q]} \{1 - \hat{K}\} \quad (77)$$

(for more details see [9]). Here the right hand side does not depend on any characteristic of the excitation and is a simple function of the magnetic field.

With the determinants in the numerator we proceed as in Appendix D of [9]. Using an idea borrowed from [3] we decomposed the measures into

$$dm_\pm^\alpha(\lambda) = \frac{d\lambda}{2\pi i} \left[ \left( \rho_n(\lambda|\alpha) \frac{1 + \alpha_0(\lambda)}{1 + \alpha_n(\lambda|\alpha)} \right)^{\pm 1} \frac{1}{1 - (\alpha_0(\lambda)/\alpha_n(\lambda|\alpha))^{\pm 1}} \right. \\ \left. + \frac{\rho_n^{\pm 1}(\lambda|\alpha)}{1 - (\alpha_n(\lambda|\alpha)/\alpha_0(\lambda))^{\pm 1}} \right]. \quad (78)$$

Then we argued that the second term in the square brackets is holomorphic inside the contours  $\mathcal{C}_n$  at least for  $T$  small enough. In fact, the functions  $\rho_n(\cdot|\alpha)^{\pm 1}$  are holomorphic inside  $\mathcal{C}_n$ . Moreover,

$$1 - \alpha_0(\lambda)/\alpha_n(\lambda|\alpha) = 1 - e^{2\pi i(w(\tilde{\lambda}) + \alpha'' - 1/2)} + \mathcal{O}(T), \quad (79)$$

where  $\tilde{\lambda} = \lambda - i\gamma/2$  and  $\lambda$  inside  $\mathcal{C}_n$ . We have numerical evidence that the  $\mathcal{O}(1)$  term on the right hand side is nonzero inside  $\mathcal{C}_n$ , where it is holomorphic as well. Since the latter is also true for the kernels, we may replace the measures  $dm_\pm^\alpha$  by the first terms on the right hand side of (78). This has the advantage that the new measures do not

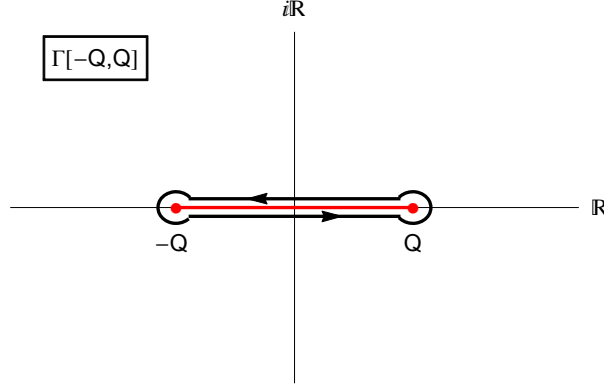


Figure 5: Integration contour  $\Gamma[-Q, Q]$  involved in the Fredholm determinants in the numerator of equation (82).

have poles that would pinch the contour at the Fermi points when  $T \rightarrow 0$ . Hence, we can shift the contour away from the Fermi points and avoid the calculation of singular integrals. Starting from (21) we then obtain

$$\rho_n(\lambda|\alpha) \frac{1 + \mathfrak{a}_0(\lambda)}{1 + \mathfrak{a}_n(\lambda|\alpha)} = e^{i\pi\alpha' + E(Q - \tilde{\lambda}) + \int_{-Q}^Q d\mu e(\mu - \tilde{\lambda})(w(\mu) + \alpha'' - 1/2)} + \mathcal{O}(T), \quad (80)$$

where  $\lambda = \tilde{\lambda} + i\gamma/2$  is outside  $\mathcal{C}_n$  and uniformly away from the Fermi points at  $\pm Q + i\gamma/2$ .

Let us define

$$\begin{aligned} d\hat{M}_{\pm}^{\alpha}(\lambda) = & \frac{d\lambda}{2\pi i} \frac{q^{\pm\alpha} e^{\pm \int_{-Q}^Q d\mu e(\mu - \lambda)(w(\mu) - w(\lambda))}}{1 - e^{\pm 2\pi i (w(\lambda) + \alpha'' - s/2)}} \\ & \times e^{\pm \{(w(\lambda) + \alpha'' + s/2)E(Q - \lambda) - (w(\lambda) + \alpha'' - s/2)E(-Q - \lambda)\}}. \end{aligned} \quad (81)$$

Then, if we perform the zero temperature limit and shift the Fermi points down to the real axis, we can replace the measures  $dm_{\pm}^{\alpha}$  in the Fredholm determinants in the numerator in (76) by  $d\hat{M}_{\pm}^{\alpha}$  with  $s = 1$ . In (81) we have also separated the factors which are singular at the Fermi points from the regular factors. Subsequently, assuming that there are no singularities between the lower part of the integration contour and the real axis, we deform the integration contour into a narrow contour  $\Gamma[-Q, Q]$  encircling the interval  $[-Q, Q]$  in positive direction (see Figure 5). Finally,

$$\mathcal{D}(p + \ell) = \frac{\det_{d\hat{M}_{+}^{\alpha}, \Gamma[-Q, Q]} \{1 - \hat{K}_{1-\alpha}\} \det_{d\hat{M}_{-}^{\alpha}, \Gamma[-Q, Q]} \{1 - \hat{K}_{1+\alpha}\}}{\det_{d\lambda/2\pi i, [-Q, Q]}^2 \{1 - \hat{K}\}}. \quad (82)$$

Here several remarks are in order. First, for the holomorphicity of the second terms on the right hand side of (78) inside  $\mathcal{C}_n$  as well as for the contraction of the contour leading to  $\Gamma[-Q, Q]$  we needed that the right hand side of (79) is non-zero inside  $\mathcal{C}_n$ . We verified this numerically with examples, but it should be justified more rigorously e.g. by establishing bounds on the imaginary part of  $w$ . Second, for  $\alpha = 0$  and  $|p + \ell|$  large

enough there may appear zeros of the right hand side of (79) in the interval  $[-Q, Q]$ . These are by definition outside the contour  $\Gamma[-Q, Q]$ .

Our third remark concerns the discontinuity of the measures across the interval  $[-Q, Q]$ . Taking into account the branch cut of  $E(-Q - \lambda)$  along the real axis from  $-Q$  to  $+\infty$  we obtain

$$\begin{aligned} d\Delta_{\pm}^{\alpha}(\lambda) &= d\hat{M}_{\pm}^{\alpha}(\lambda_{-}) - d\hat{M}_{\pm}^{\alpha}(\lambda_{+}) \\ &= \frac{d\lambda}{2\pi i} q^{\pm\alpha} e^{\pm \int_{-Q}^Q d\mu e(\mu-\lambda)(w(\mu)-w(\lambda))} e^{\pm \{ (w(\lambda)+\alpha''+s/2)E(Q-\lambda) - (w(\lambda)+\alpha''-s/2)E(-Q-\lambda_{-}) \}}. \end{aligned} \quad (83)$$

Hence, for a small  $\varepsilon > 0$ , we can interpret the integral over  $\Gamma[-Q, Q]$  with measures  $d\hat{M}_{\pm}^{\alpha}$  as a sum of an integral over  $[-Q + \varepsilon, Q - \varepsilon]$  with measures  $\Delta_{\pm}^{\alpha}$  and two integrals over infinitesimal circles of radius  $\varepsilon$  around  $-Q$  and  $Q$  with measures  $d\hat{M}_{\pm}^{\alpha}$  (see Figure 5). In general the individual contributions do not exist in the limit  $\varepsilon \rightarrow 0$ , because of the singularities of the measures at the Fermi points. For  $s = 1$  these are determined by the exponents

$$w(\pm Q) + \alpha'' \pm 1/2 = \alpha'' \mathbb{Z} \pm \frac{1}{2\mathbb{Z}}. \quad (84)$$

In the special case  $\alpha = p + \ell = 0$  which determines the leading low-temperature asymptotics (see below), however,

$$|w(\pm Q) + \alpha'' \pm 1/2| = \frac{1}{2\mathbb{Z}} < \frac{1}{\sqrt{2}}, \quad (85)$$

and the singularities of the measure are integrable. In this case we may neglect the integrals over the small circles of radius  $\varepsilon$  and replace  $d\hat{M}_{\pm}^{\alpha}$  by  $d\Delta_{\pm}^{\alpha}$  and  $\Gamma[-Q, Q]$  by  $[-Q, Q]$  in the Fredholm determinants in the numerator of (82).

Using similar ideas as above we can also obtain the zero temperature form of the integral equations (13). But when we insert (78) into the integrals in (13) we have to take into account that the functions  $\bar{G}_{\pm}(\cdot, \xi)$  are meromorphic with a single simple pole with residue  $-1$  at  $\lambda = \xi$ . For this reason the second terms on the right hand side of (78) cannot be neglected. They contribute to the driving term in the zero-temperature form of the integral equation. Another contribution is obtained when we contract the integration contours. Finally in the zero temperature limit the functions  $\bar{G}_{\pm}(\cdot, \xi)$  are determined by the integral equations

$$\begin{aligned} \bar{G}_{\pm}(\lambda, \xi) &= -\text{cth}(\lambda - \xi) \\ &+ q^{-\alpha \pm 1} \text{cth}(\lambda - \xi + \eta) q^{\pm\alpha} e^{\pm \{ E(Q - \xi + i\gamma/2) + \int_{-Q}^Q d\mu e(\mu - \xi + i\gamma/2)(w(\mu) + \alpha'' - 1/2) \}} \\ &+ \int_{\Gamma[-Q, Q]} d\hat{M}_{\pm}^{\alpha}(\mu) \bar{G}_{\pm}(\mu + i\gamma/2, \xi) K_{\alpha \mp 1}(\mu + i\gamma/2 - \lambda). \end{aligned} \quad (86)$$

Clearly the solutions depend only on  $p + \ell$ . For the physical correlation functions we will later set  $\xi = 0$ . In this case the exponential contribution to the driving term simplifies,

$$e^{\pm \{ E(Q + i\gamma/2) + \int_{-Q}^Q d\mu e(\mu + i\gamma/2)(w(\mu) + \alpha'' - 1/2) \}} = -e^{\mp 2i\alpha'' k_F}. \quad (87)$$

### 3.5 Summation

We now turn to the summation over all excitations close to the Fermi points (in the sense of (51)). An explicit summation over the particle and hole quantum numbers is possible for each value of  $\ell$ . We may use the same summation formula as employed in [20] in the context of the so-called critical form factors pertaining to the eigenstates of the ordinary transfer matrix. This formula, adapted to our notation, takes the form

$$\sum_{\substack{n_p, n_h \geq 0 \\ n_p - n_h = \ell}} \sum_{\substack{p_1 < \dots < p_{n_p} \\ p_j \in \mathbb{Z}_+}} \sum_{\substack{h_1 < \dots < h_{n_h} \\ h_j \in \mathbb{Z}_+}} e^{-\frac{2\pi m T}{v_0} [\sum_{j=1}^{n_p} (p_j - 1) + \sum_{j=1}^{n_h} h_j]} \left( \frac{\sin(\pi v)}{\pi} \right)^{2n_h} \mathcal{R}_{n_h, n_p}(\{h_j\}, \{p_j\} | v) \\ = \frac{G^2(1 + \ell - v)}{G^2(1 - v)} \frac{e^{-\frac{\pi m T \ell(\ell-1)}{v_0}}}{(1 - e^{-\frac{2\pi m T}{v_0}})^{(\ell-v)^2}}. \quad (88)$$

It can be directly applied to the results of the previous subsections. The terms that remain after the summation over the particle and hole quantum numbers depend on  $p$  and  $\ell$  only through their sum. For this reason we can shift the index and remain with a sum over  $\ell$ . Performing also the limit  $\alpha \rightarrow 0$  and setting  $\xi = 0$  we obtain the following result:

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle_{\text{osc}} = (-1)^m \sum_{\ell \in \mathbb{Z}} A_{0, \ell}^- e^{2im\ell k_F} \left( \frac{\pi T / v_0}{\text{sh}(\pi m T / v_0)} \right)^{2\ell^2 \mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}}, \quad (89)$$

where

$$A_{0, \ell}^- = \frac{e^{\frac{1}{2}\theta(2Q) + E(-2Q) + C_-[w] + C_+[w]}}{4\gamma \text{sh}(\eta)} \mathcal{D}(\ell) \overline{G}_+^-(0) \partial_\alpha \overline{G}_-^+(0) |_{\alpha=0} e^{\ell\theta(2Q) \left( \frac{\mathcal{Z}}{2} - \frac{1}{2\mathcal{Z}} \right)} \\ \times \left[ \prod_{\varepsilon_1, \varepsilon_2 = \pm 1} G \left( 1 + \varepsilon_1 \ell \mathcal{Z} + \frac{\varepsilon_2}{2\mathcal{Z}} \right) \right] \left( \frac{e^{\frac{1}{2}\theta(2Q) - E(2Q)}}{2\pi \rho(Q) \text{sh}(\eta)} \right)^{2\ell^2 \mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}}. \quad (90)$$

and where it is understood that  $\alpha = p = 0$  in those terms which depend implicitly on  $\alpha$  and  $p$ .

The series (89) is not an asymptotic series, neither in  $T$  nor in  $m$ . In each order of exponential decay we have neglected algebraic corrections in  $T$  that would contribute lower order terms than the next-order exponentials. Moreover, we have neglected higher temperature corrections to the correlation lengths that would contribute terms of the form  $\exp \mathcal{O}(mT^2)$ . The series (89) is systematic in that it gives the leading amplitudes in front of every oscillating term  $e^{2im\ell k_F}$ .

The leading low-temperature large-distance asymptotics of the transversal correlation functions is given by the  $\ell = 0$  term in the sum, *i.e.*

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle \sim \frac{e^{\frac{1}{2}\theta(2Q) + E(-2Q) + C_-[w] + C_+[w]}}{4\gamma \text{sh}(\eta)} \mathcal{D}(0) \overline{G}_+^-(0) \partial_\alpha \overline{G}_-^+(0) |_{\alpha=0} \\ \times G^2 \left( 1 + \frac{1}{2\mathcal{Z}} \right) G^2 \left( 1 - \frac{1}{2\mathcal{Z}} \right) \left( \frac{e^{\frac{1}{2}\theta(2Q) - E(2Q)}}{2\pi \rho(Q) \text{sh}(\eta)} \right)^{\frac{1}{2\mathcal{Z}^2}} (-1)^m \left( \frac{\pi T / v_0}{\text{sh}(\pi m T / v_0)} \right)^{\frac{1}{2\mathcal{Z}^2}}. \quad (91)$$

This is our main result. We shall see below that this formula is numerically efficient and matches well with known results.

## 4 Discussion

In our previous work [9] in which we derived our formulae for the amplitudes we also analyzed a generating function of the longitudinal correlation functions for small temperatures. In that work we omitted a detailed discussion of the meaning of  $p$  which was supplemented here. We also postponed the numerical analysis of the longitudinal case. Before we catch up on this let us recall the formulae.

### 4.1 A summary of the longitudinal case

In [9] we obtained an ‘oscillating series’ of similar form and meaning as (89) for a generating function of the longitudinal correlation functions,

$$\langle e^{2\pi i \alpha S(m)} \rangle_{\text{osc}} = (-1)^{m\alpha} \sum_{\ell \in \mathbb{Z}} A_{0,\ell} e^{2im(\ell-\alpha)k_F} \left( \frac{\pi T/v_0}{\text{sh}(\pi m T/v_0)} \right)^{2(\ell-\alpha)^2 \mathcal{Z}^2}. \quad (92)$$

Here  $S(m) = \sum_{j=1}^m \sigma_j^z/2$ . The amplitudes consist of two factors,  $A_{0,\ell} = \mathcal{D}_z(\ell) \mathcal{A}(\ell - \alpha)$ , where

$$\mathcal{A}(x) = e^{C_z[x\mathcal{Z}]} G^2(1+x\mathcal{Z}) G^2(1-x\mathcal{Z}) \left( \frac{e^{\frac{1}{2}\theta(2Q)-E(2Q)}}{2\pi\rho(Q)\text{sh}(\eta)} \right)^{2x^2 \mathcal{Z}^2} \quad (93)$$

with

$$C_z[v] = \frac{1}{2} \int_{-Q}^Q d\lambda \int_{-Q}^Q d\mu (v'(\lambda)v(\mu) - v(\lambda)v'(\mu)) e(\lambda - \mu) \\ + 2v(Q) \int_{-Q}^Q d\lambda (v(\lambda) - v(Q)) e(\lambda - Q). \quad (94)$$

The other factor stems from the Fredholm determinant part of the amplitudes. It is similar to (82) and can be written as

$$\mathcal{D}_z(\ell) = \frac{\det_{d\hat{M}_+^\alpha, \Gamma[-Q, Q]} \{1 - \hat{\mathcal{K}}_{-\alpha}\} \det_{d\hat{M}_-^\alpha, \Gamma[-Q, Q]} \{1 - \hat{\mathcal{K}}_\alpha\}}{\det_{\frac{d\lambda}{2\pi i}, [-Q, Q]}^2 \{1 - \hat{K}\}}. \quad (95)$$

Here it is understood that  $s = 0$  in the measures (cf. (81)) in the numerator. In this case it is convenient to absorb the factorized part of the form factors into the Fredholm determinant part [9], leading to a modification of the kernel in the Fredholm determinants in the numerator,

$$\mathcal{K}_\alpha(\lambda) = \frac{e^{(\alpha-1)(\lambda-\eta)}}{\text{sh}(\lambda-\eta)} - \frac{e^{(\alpha-1)(\lambda+\eta)}}{\text{sh}(\lambda+\eta)}. \quad (96)$$

Starting from the above oscillating series for the generating function we calculated the leading asymptotics of the longitudinal correlation functions in [9]. Taking into account only the terms with  $\ell = -1, 0, 1$  in (92) we obtained

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle - \langle \sigma_1^z \rangle \langle \sigma_{m+1}^z \rangle \sim \\ A_{0,0}^{zz} \left( \frac{\pi T/v_0}{\text{sh}(m\pi T/v_0)} \right)^2 + A_{0,1}^{zz} \cos(2mk_F) \left( \frac{\pi T/v_0}{\text{sh}(m\pi T/v_0)} \right)^{2\mathcal{Z}^2}, \quad (97a)$$

$$A_{0,0}^{zz} = -\frac{2\mathcal{Z}^2}{\pi^2}, \quad A_{0,1}^{zz} = \frac{4\sin^2(k_F)}{\pi^2} \mathcal{A}(1) \mathcal{D}_z''(1). \quad (97b)$$



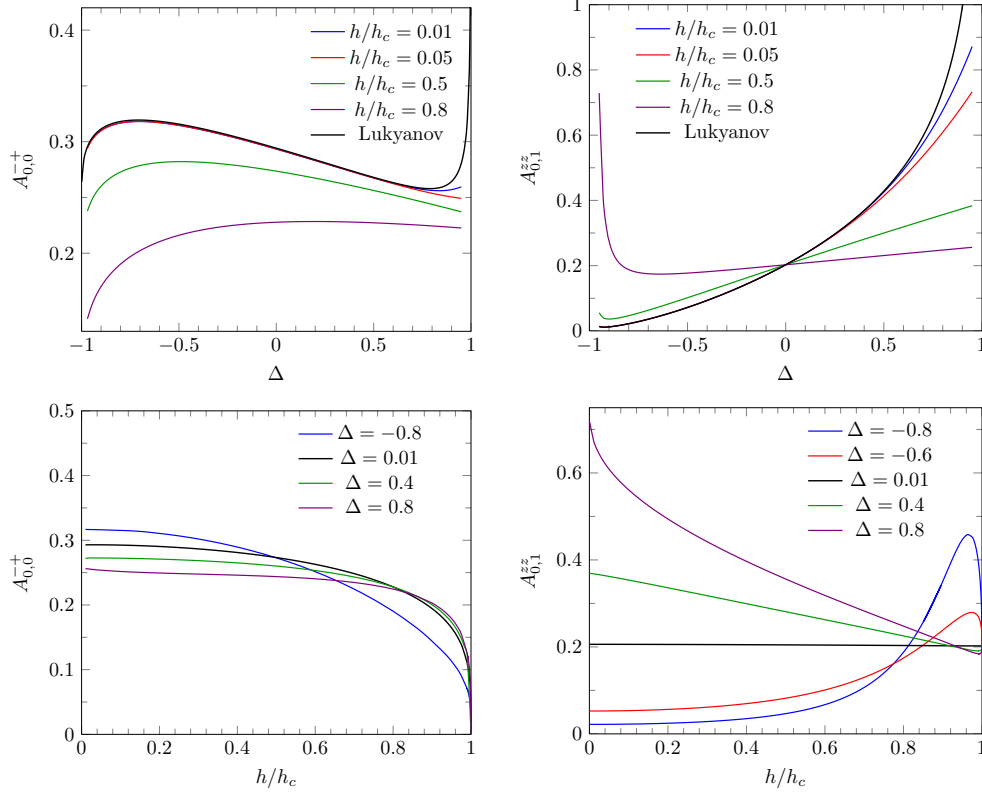


Figure 6: Amplitudes in the leading asymptotic terms as functions of the anisotropy parameter for various values of the magnetic field (upper panels) and as functions of the magnetic field for various values of the anisotropy parameter (lower panels). Transversal case in the left panels and longitudinal case in the right panels. In the longitudinal case the amplitude is leading only for  $\Delta > 0$ . For decreasing values of the magnetic field we observe numerical convergence to the  $h = 0$  result [30, 31] of Lukyanov.

Here we have used the shorthand notation  $\mathcal{D}_z''(1) = \partial_\alpha^2 \mathcal{D}_z(1)|_{\alpha=0}$ . This  $\alpha$ -derivative can be calculated analytically (see e.g. Appendix D of [9]).

## 4.2 Numerical evaluation and comparison with known results

We would like to stress that the asymptotic formulae (91) and (97) are numerically efficient and can be evaluated with standard software on a laptop computer. In fact, what has to be calculated are basically the solutions of the linear integral equations (46) and (47) which then have to be integrated over or evaluated at the Fermi points. Moreover, we have to solve the linear integral equations (86) and have to calculate the Fredholm determinants in (82). As we have learned from [6] Fredholm determinants can be efficiently calculated by discretization. The only additional problem we encounter in the Fredholm determinants in the numerator of equation (82) and also in the linear integral equations (86) is the weakly singular behaviour of the integration measures  $d\Delta_\pm^\alpha$ , equation (83). It can be dealt with by means of standard Gauß-Jacobi quadrature.

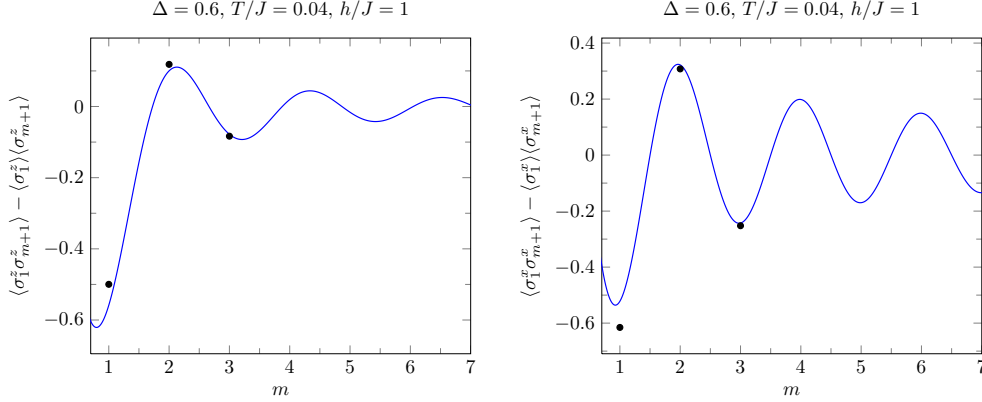


Figure 7: Correlation functions according to our asymptotic formulae (97) (left panel) and (91) (right panel). Black dots depict the exact values at short distances obtained in [1].

In Figure 6 we show the amplitudes  $A_{0,1}^{zz}$  and  $A_{0,0}^{-+}$  as functions of the anisotropy parameter  $\Delta$  for various values of the magnetic field. Recall that our formulae are valid for any positive magnetic field  $h$ . They are complementary to the  $h = 0$  results [31]

$$A_{0,1}^{zz} = \frac{8}{\pi^2} \left[ \frac{\Gamma(\frac{\pi}{2\gamma} - \frac{1}{2})}{2\sqrt{\pi}\Gamma(\frac{\pi}{2\gamma})} \right]^{\frac{\pi}{\pi-\gamma}} \exp \left\{ \int_0^\infty \frac{dk}{k} \left[ \frac{\text{sh}((1 - \frac{2\gamma}{\pi})k)}{\text{sh}((1 - \frac{\gamma}{\pi})k) \text{ch}(\frac{\gamma k}{\pi})} - \left(1 - \frac{\gamma}{\pi - \gamma}\right) e^{-2k} \right] \right\},$$

$$A_{0,0}^{-+} = \frac{\pi^2}{4\gamma^2} \left[ \frac{\Gamma(\frac{\pi}{2\gamma} - \frac{1}{2})}{2\sqrt{\pi}\Gamma(\frac{\pi}{2\gamma})} \right]^{1 - \frac{\gamma}{\pi}} \exp \left\{ \int_0^\infty \frac{dk}{k} \left[ \left(1 - \frac{\gamma}{\pi}\right) e^{-2k} - \frac{\text{sh}((1 - \frac{\gamma}{\pi})k)}{\text{sh}(k) \text{ch}(\frac{\gamma k}{\pi})} \right] \right\} \quad (98)$$

obtained in a quantum field theoretic setting [7, 30, 32, 33] starting from the Gaussian model and taking into account the most relevant irrelevant operators. The amplitudes (98) are plotted in black in the upper panels of Figure 6. Clearly, our field-dependent amplitudes numerically converge to these amplitudes for  $h \rightarrow 0$ . The discrepancies close to  $\Delta = 1$  are not numerical artifacts. They rather indicate the highly singular behaviour of the amplitudes close to this point. So far we do not know how to obtain the expressions (98) directly from our formulae (97) and (91). In the lower panels of Figure 6 we show the field dependence of the amplitudes. Their slope as functions of  $h$  becomes infinite at the critical field  $h_c$ , where the phase transition to the fully polarized phase occurs.

Knowing the amplitudes we can plot the correlation functions as functions of the distance. This is shown in Figure 7, where we also compare the asymptotic behaviour with the exact short-distance behaviour of the two-point functions as obtained in [1]. As we see, the asymptotic formulae provide rather accurate approximations down to the smallest possible distance  $m = 1$ . This seems less amazing if we take into account that fairly good agreement between the leading order asymptotic formulae and the exact short-distance results was even obtained at the isotropic point, where logarithmic corrections are important [38].

This encouraged us to compare the magnetic field dependence of the third-neighbour correlation functions as obtained from the asymptotic formulae with the exact results,

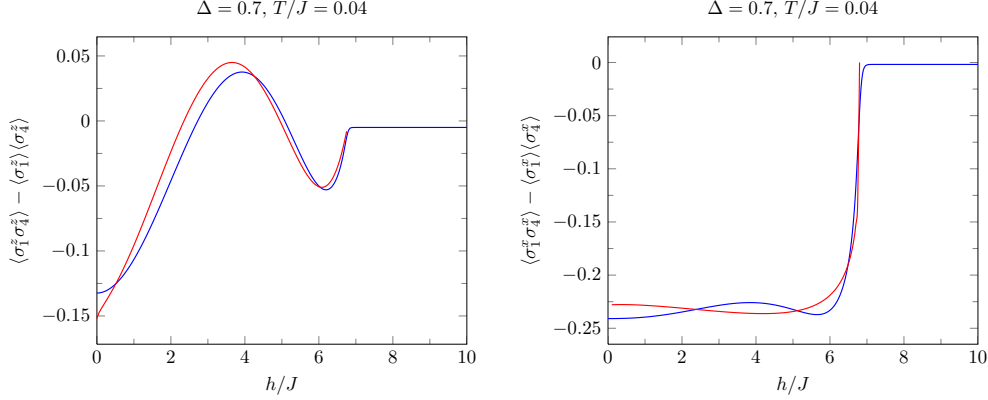


Figure 8: Comparison of magnetic field dependence of the third-neighbour correlators, exact [1] blue lines, asymptotic red lines. Longitudinal case in the left panel, transversal case in the right panel.

which is shown in Figure 8. Again the agreement is good. It becomes worse if we approach the isotropic point. The temperature behaviour of the third-neighbour longitudinal two-point function for two different values of anisotropy is shown in Figure 9. For negative  $\Delta$ , where these correlation functions exhibit a ‘quantum classical crossover’ [1, 10, 11], the asymptotic result ceases to be a good approximation for temperatures above the crossover temperature. For positive  $\Delta$ , however, where no such crossover occurs, the asymptotic formula provides a reasonably accurate description of the correlation functions up to arbitrary temperatures.

## 5 Conclusion

In this work we have continued the low-temperature analysis of the correlation lengths and amplitudes occurring in the form factor expansion of the two-point correlation functions of the spin-1/2 XXZ chain. We started our analysis with expressions for the amplitudes which were obtained in [9], where we combined algebraic Bethe ansatz methods for the calculation of form factors with the quantum transfer matrix approach to thermodynamics [41, 42] and with the method of nonlinear integral equations [24, 25]. Similar formulae hold for finite-size systems at zero temperature. In both cases the formulae are exact up to this stage. Their low-temperature (or finite-size) analysis is a logically independent task. Here we dealt with the low-temperature case for  $|\Delta| < 1$  and positive magnetic field. We provided, in particular, an extensive discussion of the low-temperature behaviour of the solution of the fundamental nonlinear integral equations. Higher temperatures and the case  $\Delta > 1$  will be addressed in separate works.

Based on the low-temperature analysis of the nonlinear integral equations we obtained the leading low-temperature expressions for the correlation lengths and for the amplitudes associated with those excitations that ‘collapse to the Fermi points’ for  $T \rightarrow 0$ . These amplitudes show ‘critical behaviour’: they vanish as fractional powers of the temperature with critical exponents determined by the scaling dimensions of the underlying conformal field theory. Using a summation formula obtained in [16, 20, 34], we could sum up the corresponding terms in the form factor series and obtained

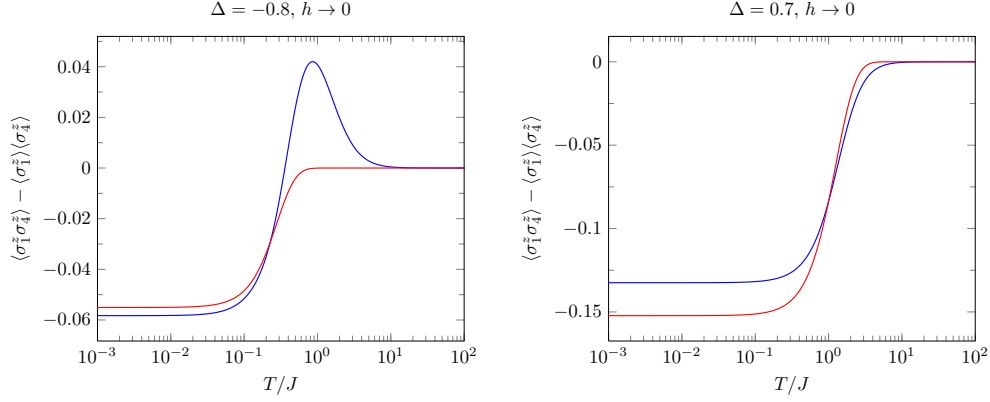


Figure 9: Comparison of temperature dependence of third-neighbour correlators, exact [1] blue lines, asymptotic red lines. The sign change of the correlation function for negative  $\Delta$  (blue line in left panel) has been explained by a ‘quantum classical crossover’ [1, 10, 11]. Accordingly it is not seen in our low-temperature asymptotic expansion which covers only the ‘quantum regime’ of the phase diagram.

the leading low-temperature large-distance asymptotics of the transversal two-point functions. A similar asymptotic formula for the longitudinal case was obtained in [9]. Our formulae include explicit expressions for the amplitudes for any positive magnetic field which do not follow directly from conformal field theory and are complementary to Lukyanov’s formulae [31] for  $h = 0$ . Similar but different expressions for the amplitudes were obtained previously in the context of scaling analysis for large system size at  $T = 0$  [18, 21, 39]. Our formulae have turned out to be numerically efficient. They can be evaluated on a laptop computer and the resulting curves for amplitudes and correlation functions match well with known results.

**Acknowledgment.** The authors are grateful to A. Klümper, J. Suzuki and A. Weiße for helpful discussions and encouragement. The numerical data for the exact short-range correlation functions were generously provided by M. Brockmann. MD and FG acknowledge financial support by the Volkswagen Foundation and by the Deutsche Forschungsgemeinschaft under grant number Go 825/7-1. KKK is supported by the CNRS. His work has been partly financed by the grant PEPS-PTI ‘Asymptotique d’intégrales multiples’ and by a Burgundy region PARI 2013 FABER grant ‘Structures et asymptotiques d’intégrales multiples’.

## Appendix A: Two proofs

In this appendix we would like to provide the proofs of Lemma 2 and Lemma 3 that were left out in the main text.

### A.1 Proof of Lemma 2

By the same reasoning as in the proof of Lemma 1 we obtain for any  $\lambda_{\pm}$  not on  $\mathcal{C}_u$

$$\begin{aligned}\Delta I_u(\lambda_{\pm}) &:= I_u(\lambda_{\pm}) + \int_{Q_-}^{Q_+} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \frac{\bar{u}(\lambda)}{T} \\ &= \int_{\mathcal{C}_u} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \ln \left( 1 + e^{-\frac{u(\lambda) \operatorname{sign}(v(\lambda))}{T}} \right) \\ &= \int_{J_-^{\delta} \cup J_+^{\delta}} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \ln \left( 1 + e^{-\frac{|v(\lambda)|}{T}} \right) + \mathcal{O}(T^{\infty}),\end{aligned}\quad (\text{A.1})$$

where  $v(\lambda) = \operatorname{Re} u(\lambda)$ , and  $\delta > 0$  and  $J_{\pm}^{\delta}$  are chosen as in the proof of Lemma 1.

If  $\lambda_{\pm} \in V_+$ , then

$$\begin{aligned}&\int_{J_-^{\delta} \cup J_+^{\delta}} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \ln \left( 1 + e^{-\frac{|v(\lambda)|}{T}} \right) \\ &= \int_{J_+^{\delta}} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \ln \left( 1 + e^{-\frac{|v(\lambda)|}{T}} \right) + \mathcal{O}(T) \\ &= \int_{J_+^{\delta}} d\lambda \left[ \operatorname{cth}(\lambda - \lambda_{\pm}) - \frac{u'(\lambda)}{u(\lambda) - u(\lambda_{\pm})} \right] \ln \left( 1 + e^{-\frac{|v(\lambda)|}{T}} \right) \\ &\quad + \int_{J_+^{\delta}} d\lambda \frac{v'(\lambda)}{v(\lambda) + iw(\lambda) - u(\lambda_{\pm})} \ln \left( 1 + e^{-\frac{|v(\lambda)|}{T}} \right) + \mathcal{O}(T) \\ &= \int_{-\delta/T}^{\delta/T} dx \frac{\ln(1 + e^{-|x|})}{x - \bar{u}(\lambda_{\pm})/T} + \mathcal{O}(T).\end{aligned}\quad (\text{A.2})$$

Here we used (37) in the first equation and a similar identity with  $J_+^{\delta}$  replacing  $J_-^{\delta}$  in the third equation. We further employed the fact that  $w(\lambda) = 2\pi i T$  on  $J_+^{\delta}$  in the second and third equation. As in the proof of Lemma 1 we substituted  $x = v(\lambda)/T$ .

For  $\lambda_{\pm}$  in the vicinity of  $Q_+$  we cannot assume that  $a_{\pm} := \bar{u}(\lambda_{\pm})/T$  is large for small  $T$ . We therefore treat  $a_{\pm}$  as an independent parameter. Then (A.1) and (A.2) imply that

$$\Delta I_u(\lambda_{\pm}) = \int_{-\infty}^{\infty} dx \frac{\ln(1 + e^{-|x|})}{x - a_{\pm}} + \mathcal{O}(T).\quad (\text{A.3})$$

Note that  $a_+$  is in the upper half plane and  $a_-$  is in the lower half plane. The integral on the right hand side can be calculated by means of the residue theorem (for a similar calculation in the context of the Bose gas with delta function interaction see [28]),

$$\begin{aligned}&\int_{-\infty}^{\infty} dx \frac{\ln(1 + e^{-|x|})}{x - a_{\pm}} \\ &= \mp 2\pi i \ln \left\{ \Gamma \left( \frac{1}{2} \pm \frac{a_{\pm}}{2\pi i} \right) \right\} \pm \pi i \ln(2\pi) + a_{\pm} \ln \left( \frac{a_{\pm}}{\pm 2\pi i} \right) - a_{\pm}.\end{aligned}\quad (\text{A.4})$$

On the other hand

$$\begin{aligned} \int_{Q_-}^{Q_+} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \frac{\bar{u}(\lambda)}{T} \\ = \int_{Q_-}^{Q_+} d\lambda \operatorname{cth}(\lambda - \lambda_{\pm}) \frac{\bar{u}(\lambda) - \bar{u}(\lambda_{\pm})}{T} + \frac{\bar{u}(\lambda_{\pm})}{T} \ln \left( \frac{\operatorname{sh}(Q_+ - \lambda_{\pm})}{\operatorname{sh}(Q_- - \lambda_{\pm})} \right), \end{aligned} \quad (\text{A.5})$$

and (72) follows from (A.1)-(A.5). The proof of (73) is similar and is left to the reader.

## A.2 Proof of Lemma 3

In preparation of the proof we shall need some results which are either easy to see or were proved elsewhere. First of all we have the following corollary of Lemma 1 and Lemma 2.

**Corollary 1.** *Let  $\lambda \in V_{\pm}$  and  $\sigma = \operatorname{sign}(\operatorname{Im} \lambda)$ . Then  $\lambda$  is above  $\mathbb{C}_{0,s} - i\gamma/2$  if  $\sigma > 0$ , below  $\mathbb{C}_{0,s} - i\gamma/2$  if  $\sigma < 0$ , provided that  $T$  is small enough, and*

$$\begin{aligned} L_{\mathbb{C}_{0,s} - i\gamma/2}[z](\lambda) &= \int_{\mathbb{C}_{0,s} - i\gamma/2} d\mu \operatorname{cth}(\mu - \lambda) z(\mu + i\gamma/2) \\ &= - \int_{-Q}^Q \frac{d\mu}{2\pi i} \operatorname{cth}(\mu - \lambda) (\bar{u}_1(\mu) - \bar{u}_1(\lambda)) + \frac{\bar{u}_1(\lambda)}{2\pi i} \ln \left( \frac{\varepsilon^{\pm 1}(\lambda) \operatorname{sh}(\lambda + Q)}{\operatorname{sh}(\lambda - Q)} \right) \\ &\quad \mp \frac{\bar{u}_1(\lambda)}{2\pi i} \ln(\pm \sigma 2\pi i T) - \sigma \ln \left( \frac{\Gamma(1/2 \pm \sigma \bar{u}(\lambda)/2\pi i T)}{\Gamma(1/2 \pm \sigma \bar{u}_0(\lambda)/2\pi i T)} \right) + \mathcal{O}(T). \end{aligned} \quad (\text{A.6})$$

On the other hand, if  $\lambda$  is uniformly away from  $\pm Q$ , it follows that

$$L_{\mathbb{C}_{0,s} - i\gamma/2}[z](\lambda) = - \int_{-Q}^Q \frac{d\mu}{2\pi i} \operatorname{cth}(\mu - \lambda) \bar{u}_1(\mu) + \mathcal{O}(T). \quad (\text{A.7})$$

The following useful lemma was proved in Appendix B of [19].

**Lemma 4.** *Let  $I \subset \mathbb{R}$  be an open interval containing 0. Let  $\mathcal{R} : I \times \mathbb{R}_+ \rightarrow \mathbb{C}$  such that  $u \rightarrow \mathcal{R}(u, t)$  is  $\mathcal{C}^1(I)$  for all but finitely many  $t$  and  $t \rightarrow \mathcal{R}(u, t)$  is Riemann integrable uniformly in  $u$ , i.e.,  $\forall \varepsilon > 0, \forall M > 0, \forall u_0 \in I \exists \delta > 0$  such that  $u \in (u_0 - \delta, u_0 + \delta) \cap I \Rightarrow$*

$$\left| \int_M^\infty dt (\partial_1^k \mathcal{R}(u, t) - \partial_1^k \mathcal{R}(u_0, t)) \right| < \varepsilon \quad (\text{A.8})$$

for  $k = 0, 1$ .

Then, for  $g \in \mathcal{C}^1(I)$ ,

$$\int_0^\delta dt x g(t) \mathcal{R}(t, xt) = g(0) \int_0^\infty dt \mathcal{R}(0, t) + o(1), \quad (\text{A.9})$$

where  $o(1)$  denotes terms that vanish in the ordered limit first  $x\delta \rightarrow \infty$  and then  $\delta \rightarrow 0$ .

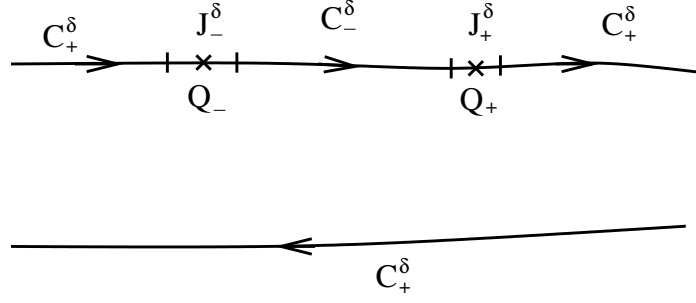


Figure 10: Deformation and decomposition of the contour  $-(\mathcal{C}_{0,s} - i\gamma/2)$ .

Now, for the proof of Lemma 3, we rewrite the double integral as

$$A = - \int_{\mathcal{C}_{0,s}} d\lambda \int_{\mathcal{C}'_{0,s}} d\mu z(\lambda) \text{cth}'(\lambda - \mu) z(\mu) = \int_{-(\mathcal{C}_{0,s} - i\gamma/2)} \frac{d\lambda}{2\pi i T} \left[ \frac{u'(\lambda)}{1 + e^{u(\lambda)/T}} - \frac{u'_0(\lambda)}{1 + e^{u_0(\lambda)/T}} \right] L_{\mathcal{C}'_{0,s} - i\gamma/2}[z](\lambda). \quad (\text{A.10})$$

Then we deform the contour  $-(\mathcal{C}_{0,s} - i\gamma/2)$  and decompose it into four disjoint pieces,  $-(\mathcal{C}_{0,s} - i\gamma/2) \rightarrow \mathcal{C}_+^\delta \cup \mathcal{C}_-^\delta \cup J_+^\delta \cup J_-^\delta$  (see Figure 10). Here  $\delta > 0$ , and  $J_\pm^\delta$  are chosen as in the proof of Lemma 1. They contain  $Q_\pm$ ,  $\text{Im } u = 2\pi pT$  in  $J_\pm^\delta$  and  $\text{Re } u$  grows (decreases) monotonically from  $-\delta$  to  $\delta$  in  $J_+^\delta$  ( $J_-^\delta$ ). Moreover,  $\text{Re } u > 0$  on  $\mathcal{C}_+^\delta$ ,  $\text{Re } u < 0$  on  $\mathcal{C}_-^\delta$ . If  $T$  is sufficiently small the latter will be also true for  $\text{Re } u_0$ . It follows that

$$A = I_{\mathcal{C}_-^\delta} + I_{J_-^\delta} + I_{J_+^\delta} + \mathcal{O}(T^\infty), \quad (\text{A.11})$$

where

$$I_{\mathcal{C}} = \int_{\mathcal{C}} \frac{d\lambda}{2\pi i T} \left[ \frac{u'(\lambda)}{1 + e^{u(\lambda)/T}} - \frac{u'_0(\lambda)}{1 + e^{u_0(\lambda)/T}} \right] L_{\mathcal{C}_{0,s} - i\gamma/2}[z](\lambda_+) \quad (\text{A.12})$$

and  $\lambda_+$  is the boundary value from above.

Using Corollary 1 we obtain

$$\begin{aligned} I_{\mathcal{C}_-^\delta} &= \int_{\mathcal{C}_-^\delta} \frac{d\lambda}{2\pi i T} (u'(\lambda) - u'_0(\lambda)) L_{\mathcal{C}_{0,s} - i\gamma/2}[z](\lambda_+) + \mathcal{O}(T^\infty) \\ &= - \int_{\mathcal{C}_-^\delta} \frac{d\lambda}{2\pi i} u'_1(\lambda) \int_{-Q}^Q \frac{d\mu}{2\pi i} \text{cth}(\mu - \lambda_+) \bar{u}_1(\mu) + \mathcal{O}(T) \\ &= - \int_{-Q}^Q \frac{d\lambda}{2\pi i} u'_1(\lambda) \int_{-Q}^Q \frac{d\mu}{2\pi i} \text{cth}(\mu - \lambda_+) \bar{u}_1(\mu) + o(1) \\ &= \int_{-Q}^Q \frac{d\lambda}{2\pi i} \int_{-Q}^Q \frac{d\mu}{2\pi i} \frac{\bar{u}'_1(\lambda) \bar{u}_1(\mu) - \bar{u}'_1(\mu) \bar{u}_1(\lambda)}{2 \text{th}(\lambda - \mu)} \\ &\quad - \frac{1}{8\pi i} (\bar{u}_1^2(Q) - \bar{u}_1^2(-Q)) + o(1), \end{aligned} \quad (\text{A.13})$$

where  $o(1)$  denotes terms that vanish in the ordered limit first  $\delta/T \rightarrow \infty$  and then  $\delta \rightarrow 0$ .

In order to evaluate the remaining integrals we split them further. Setting  $\delta u = (\bar{u} - u_0)/T$  we shall write  $I_{J_{\pm}^{\delta}} = I_{J_{\pm}^{\delta}}^{(1)} + I_{J_{\pm}^{\delta}}^{(2)}$ , where

$$I_{J_{\pm}^{\delta}}^{(1)} = \int_{J_{\pm}^{\delta}} \frac{d\lambda}{2\pi i T} u'(\lambda) \left[ \frac{1}{1 + e^{\bar{u}(\lambda)/T}} - \frac{1}{1 + e^{\bar{u}(\lambda)/T - \delta u(\lambda)}} \right] L_{\mathcal{C}_{0,s} - i\gamma/2}[z](\lambda_+), \quad (\text{A.14a})$$

$$I_{J_{\pm}^{\delta}}^{(2)} = \int_{J_{\pm}^{\delta}} \frac{d\lambda}{2\pi i} \frac{\delta u'(\lambda)}{1 + e^{\bar{u}(\lambda)/T - \delta u(\lambda)}} L_{\mathcal{C}_{0,s} - i\gamma/2}[z](\lambda_+). \quad (\text{A.14b})$$

Recall that  $\bar{u}$  maps  $J_{\pm}^{\delta}$  to  $[-\delta, \delta]$  in such a way that  $\bar{u}(Q_{\pm}) = 0$ . Hence, the substitution  $t = \bar{u}(\lambda)$  transforms  $I_{J_{+}^{\delta}}$  into an integral of the form appearing on the left hand side of (A.9). Since the requirements of Lemma 4 are satisfied, we conclude that

$$\begin{aligned} I_{J_{+}^{\delta}}^{(1)} &= \int_{-\infty}^{\infty} \frac{dt}{2\pi i} \left[ \frac{1}{1 + e^t} - \frac{1}{1 + e^{t - \bar{u}_1(Q)}} \right] \left\{ - \int_{-Q}^Q \frac{d\mu}{2\pi i} \text{cth}(\mu - Q) (\bar{u}_1(\mu) - \bar{u}_1(Q)) \right. \\ &\quad \left. - \frac{\bar{u}_1(Q)}{2\pi i} \ln \left( \frac{2\pi i T}{\epsilon'(Q) \text{sh}(2Q)} \right) - \ln \left( \frac{\Gamma(1/2 + t/2\pi i)}{\Gamma(1/2 + t/2\pi i - \bar{u}_1(Q)/2\pi i)} \right) \right\} + o(1) \\ &= \frac{\bar{u}_1(Q)}{2\pi i} \int_{-Q}^Q \frac{d\mu}{2\pi i} \text{cth}(\mu - Q) (\bar{u}_1(\mu) - \bar{u}_1(Q)) + \left( \frac{\bar{u}_1(Q)}{2\pi i} \right)^2 \ln \left( \frac{2\pi i T}{\epsilon'(Q) \text{sh}(2Q)} \right) \\ &\quad + \ln \left\{ G \left( 1 - \frac{\bar{u}_1(Q)}{2\pi i} \right) G \left( 1 + \frac{\bar{u}_1(Q)}{2\pi i} \right) \right\} + o(1). \quad (\text{A.15}) \end{aligned}$$

Here we have used Corollary 1 and dropped  $\mathcal{O}(T)$  corrections in the integrand in the first equation. The remaining integrals have been calculated by means of residue calculus (cf. Appendix B of [28]). A similar calculation yields

$$\begin{aligned} I_{J_{-}^{\delta}}^{(1)} &= - \frac{\bar{u}_1(-Q)}{2\pi i} \int_{-Q}^Q \frac{d\mu}{2\pi i} \text{cth}(\mu + Q) (\bar{u}_1(\mu) - \bar{u}_1(-Q)) \\ &\quad + \left( \frac{\bar{u}_1(-Q)}{2\pi i} \right)^2 \ln \left( \frac{-2\pi i T}{\epsilon'(Q) \text{sh}(2Q)} \right) \\ &\quad + \ln \left\{ G \left( 1 + \frac{\bar{u}_1(-Q)}{2\pi i} \right) G \left( 1 - \frac{\bar{u}_1(-Q)}{2\pi i} \right) \right\} + o(1). \quad (\text{A.16}) \end{aligned}$$

The remaining integrals will turn out to be  $o(1)$ . In order to prove this assertion we split them once more into  $I_{J_{\pm}^{\delta}}^{(2)} = I_{J_{\pm}^{\delta}}^{(3)} + I_{J_{\pm}^{\delta}}^{(4)}$ , where

$$I_{J_{\pm}^{\delta}}^{(3)} = \int_{J_{\pm}^{\delta}} \frac{d\lambda}{2\pi i} \delta u'(\lambda) \left[ \frac{1}{1 + e^{\bar{u}(\lambda)/T - \delta u(\lambda)}} - \Theta \left( -\frac{\bar{u}(\lambda)}{T} \right) \right] L_{\mathcal{C}_{0,s} - i\gamma/2}[z](\lambda_+), \quad (\text{A.17a})$$

$$I_{J_{\pm}^{\delta}}^{(4)} = \int_{J_{\pm}^{\delta}} \frac{d\lambda}{2\pi i} \delta u'(\lambda) \Theta \left( -\frac{\bar{u}(\lambda)}{T} \right) L_{\mathcal{C}_{0,s} - i\gamma/2}[z](\lambda_+). \quad (\text{A.17b})$$

Here  $\Theta$  is the Heaviside step function. The integrals  $I_{J_{\pm}^{\delta}}^{(3)}$  can be treated as  $I_{J_{\pm}^{\delta}}^{(1)}$ . Again Lemma 4 applies, and the resulting terms are  $o(1)$ .



In order to estimate  $I_{J_+^\delta}^{(4)}$  we write it as

$$I_{J_+^\delta}^{(4)} = \int_{Q_+^-}^{Q_+} \frac{d\lambda}{2\pi i} \bar{u}'_1(\lambda) \left\{ -\frac{\bar{u}_1(\lambda)}{2\pi i} \ln(2\pi i T) - \ln \left( \frac{\Gamma(1/2 + \bar{u}(\lambda)/2\pi i T)}{\Gamma(1/2 + \bar{u}(\lambda)/2\pi i T - \bar{u}_1(\lambda)/2\pi i)} \right) \right\} + o(1). \quad (\text{A.18})$$

Here we have denoted the left end point of  $J_+^\delta$ , where  $\bar{u} = -\delta$ , by  $Q_+^-$ . We have inserted (A.6) in which we have suppressed the  $\mathcal{O}(1)$  terms. Now the integral over the first term in curly brackets can be evaluated explicitly. In order to estimate the second term we note that due to Stirling's formula

$$\ln \left( \frac{\Gamma(1/2 + x)}{\Gamma(1/2 + x - a)} \right) - a \ln(1 + x) + \frac{a + a^2/2}{1 + x} = \mathcal{O}(1/x^2). \quad (\text{A.19})$$

Hence, if we rewrite  $I_{J_+^\delta}^{(4)}$  as

$$\begin{aligned} I_{J_+^\delta}^{(4)} &= \frac{\bar{u}_1^2(Q_+^-) - \bar{u}_1^2(Q_+)}{8(\pi i)^2} \ln(2\pi i T) \\ &\quad - \int_{Q_+^-}^{Q_+} \frac{d\lambda}{2\pi i} \bar{u}'_1(\lambda) \left\{ \ln \left( \frac{\Gamma(1/2 + \bar{u}(\lambda)/2\pi i T)}{\Gamma(1/2 + \bar{u}(\lambda)/2\pi i T - \bar{u}_1(\lambda)/2\pi i)} \right) \right. \\ &\quad \left. - \frac{\bar{u}_1(\lambda)}{2\pi i} \ln \left( 1 + \frac{\bar{u}(\lambda)}{2\pi i T} \right) + \frac{\bar{u}_1(\lambda)}{2\pi i} \left( 1 + \frac{\bar{u}_1(\lambda)}{4\pi i} \right) \left[ 1 + \frac{\bar{u}(\lambda)}{2\pi i T} \right]^{-1} \right\} \\ &\quad - \int_{Q_+^-}^{Q_+} \frac{d\lambda}{2\pi i} \bar{u}'_1(\lambda) \left\{ \frac{\bar{u}_1(\lambda)}{2\pi i} \ln \left( 1 + \frac{\bar{u}(\lambda)}{2\pi i T} \right) - \frac{\bar{u}_1(\lambda)}{2\pi i} \left( 1 + \frac{\bar{u}_1(\lambda)}{4\pi i} \right) \left[ 1 + \frac{\bar{u}(\lambda)}{2\pi i T} \right]^{-1} \right\} \\ &\quad + o(1), \quad (\text{A.20}) \end{aligned}$$

then, after substituting  $x = \bar{u}(\lambda)$ , the integrand in the first integral satisfies the requirements of Lemma 4, and the integral turns out to be  $o(1)$ . The second term in the second integral is  $\mathcal{O}(T)$  and can be neglected. It follows that

$$\begin{aligned} I_{J_+^\delta}^{(4)} &= \frac{\bar{u}_1^2(Q_+^-) - \bar{u}_1^2(Q_+)}{8(\pi i)^2} \ln(2\pi i T) - \int_{Q_+^-}^{Q_+} \frac{d\lambda}{2\pi i} \bar{u}'_1(\lambda) \frac{\bar{u}_1(\lambda)}{2\pi i} \ln \left( 1 + \frac{\bar{u}(\lambda)}{2\pi i T} \right) + o(1) \\ &= \frac{\bar{u}_1^2(Q_+^-) - \bar{u}_1^2(Q_+)}{8(\pi i)^2} \ln(-\delta) + o(1) = o(1). \quad (\text{A.21}) \end{aligned}$$

In order to estimate the remaining integral we performed a partial integration and used that

$$\int_{-\delta}^0 \frac{dt}{2\pi i} \frac{g(t)x}{1 + tx/2\pi i} = -g(0) \ln \left( -\frac{x\delta}{2\pi i} \right) + o(1), \quad (\text{A.22})$$

where  $g \in \mathcal{C}^1(I)$  for an open interval  $I$  containing 0, and  $o(1)$  denotes terms which vanish in the ordered limit first  $x\delta \rightarrow \infty$  and then  $\delta \rightarrow 0$  (cf. Appendix B of [19]). The integral  $I_{J_-^\delta}^{(4)}$  can be treated in a similar way and turns out to be  $o(1)$  as well.

Summarizing the above we have shown that  $A = I_{\mathbb{C}_-^\delta} + I_{J_-^\delta}^{(1)} + I_{J_+^\delta}^{(1)} + o(1)$ . Then Lemma 3 follows from (A.13), (A.15) and (A.16).

## References

- [1] H. Boos, J. Damerau, F. Göhmann, A. Klümper, J. Suzuki, and A. Weiße, *Short-distance thermal correlations in the XXZ chain*, J. Stat. Mech.: Theor. Exp. (2008), P08010.
- [2] H. Boos and F. Göhmann, *On the physical part of the factorized correlation functions of the XXZ chain*, J. Phys. A **42** (2009), 315001.
- [3] ———, *Properties of linear integral equations related to the six-vertex model with disorder parameter*, New Trends in Quantum Integrable Systems (Boris Feigin, Michio Jimbo, and Masato Okado, eds.), World Scientific, Singapore, 2010, Proceedings of the conference on Infinite Analysis 09, pp 1-10.
- [4] H. Boos, F. Göhmann, A. Klümper, and J. Suzuki, *Factorization of the finite temperature correlation functions of the XXZ chain in a magnetic field*, J. Phys. A **40** (2007), 10699.
- [5] H. Boos, M. Jimbo, T. Miwa, and F. Smirnov, *Hidden Grassmann structure in the XXZ model IV: CFT limit*, Comm. Math. Phys. **299** (2010), 825.
- [6] F. Bornemann, *On the numerical evaluation of Fredholm determinants*, Mathematics of Computation **79** (2010), 871.
- [7] J. L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nucl. Phys. B **270** (1986), 186.
- [8] M. Dugave, F. Göhmann, and K. K. Kozłowski, *Functions characterizing the ground state of the XXZ spin-1/2 chain in the thermodynamic limit*, preprint arXiv:1311.6959, 2013.
- [9] ———, *Thermal form factors of the XXZ chain and the large-distance asymptotics of its temperature dependent correlation functions*, J. Stat. Mech.: Theor. Exp. (2013), P07010.
- [10] K. Fabricius, A. Klümper, and B. M. McCoy, *Temperature-dependent spatial oscillations in the correlations of the XXZ spin chain*, Phys. Rev. Lett. **82** (1999), 5365.
- [11] K. Fabricius and B. M. McCoy, *Quantum-classical crossover in the spin-1/2 XXZ chain*, Phys. Rev. B **59** (1999), 381.
- [12] F. Göhmann, A. Klümper, and A. Seel, *Integral representations for correlation functions of the XXZ chain at finite temperature*, J. Phys. A **37** (2004), 7625.
- [13] ———, *Integral representation of the density matrix of the XXZ chain at finite temperature*, J. Phys. A **38** (2005), 1833.
- [14] M. Jimbo and T. Miwa, *Quantum KZ equation with  $|q| = 1$  and correlation functions of the XXZ model in the gapless regime*, J. Phys. A **29** (1996), 2923.

- [15] M. Jimbo, T. Miwa, and F. Smirnov, *Hidden Grassmann structure in the XXZ model III: introducing Matsubara direction*, J. Phys. A **42** (2009), 304018.
- [16] S. Kerov, G. Olshanski, and A. Vershik, *Harmonic analysis on the infinite symmetric group. A deformation of the regular representation*, Compt. Rend. Acad. Sci. Paris, Ser. I **316** (1993), 773.
- [17] N. Kitanine, K. K. Kozłowski, J. M. Maillet, N. A. Slavnov, and V. Terras, *Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions*, J. Stat. Mech.: Theor. Exp. (2009), P04003.
- [18] ———, *On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain*, J. Math. Phys. **50** (2009), 095209.
- [19] ———, *Riemann-Hilbert approach to a generalised sine kernel and applications*, Comm. Math. Phys. **291** (2009), 691.
- [20] ———, *A form factor approach to the asymptotic behavior of correlation functions in critical models*, J. Stat. Mech.: Theor. Exp. (2011), P12010.
- [21] ———, *The thermodynamic limit of particle-hole form factors in the massless XXZ Heisenberg chain*, J. Stat. Mech.: Theor. Exp. (2011), P05028.
- [22] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *Master equation for spin-spin correlation functions of the XXZ chain*, Nucl. Phys. B **712** (2005), 600.
- [23] N. Kitanine, J. M. Maillet, and V. Terras, *Form factors of the XXZ Heisenberg spin- $\frac{1}{2}$  finite chain*, Nucl. Phys. B **554** (1999), 647.
- [24] A. Klümper, *Free energy and correlation length of quantum chains related to restricted solid-on-solid lattice models*, Ann. Physik **1** (1992), 540.
- [25] ———, *Thermodynamics of the anisotropic spin-1/2 Heisenberg chain and related quantum chains*, Z. Phys. B **91** (1993), 507.
- [26] A. Klümper and C. Scheeren, *The thermodynamics of the spin-1/2 XXX chain: free energy and low-temperature singularities of correlation lengths*, Classical and Quantum Nonlinear Integrable Systems (A. Kundu, ed.), Series in Mathematical and Computational Physics, IOP publishing, Bristol, 2003, pp. 234–255.
- [27] V. E. Korepin and N. A. Slavnov, *The new identity for the scattering matrix of exactly solvable models*, Eur. Phys. J. B **5** (1998), 555.
- [28] K. K. Kozłowski, J. M. Maillet, and N. A. Slavnov, *Correlation functions for one-dimensional bosons at low temperature*, J. Stat. Mech.: Theor. Exp. (2011), P03019.
- [29] ———, *Long-distance behavior of temperature correlation functions in the one-dimensional Bose gas*, J. Stat. Mech.: Theor. Exp. (2011), P03018.
- [30] S. Lukyanov, *Low energy effective Hamiltonian for the XXZ chain*, Nucl. Phys. B **522** (1998), 533.

- [31] ———, *Correlation amplitude for the XXZ spin chain in the disordered regime*, Phys. Rev. B **59** (1999), 11163.
- [32] S. Lukyanov and V. Terras, *Long-distance asymptotics of spin-spin correlation functions for the XXZ spin chain*, Nucl. Phys. B **654** (2003), 323.
- [33] A. Luther and I. Peschel, *Calculation of critical exponents in two dimensions from quantum field theory in one dimension*, Phys. Rev. B **12** (1975), 3908.
- [34] G. Olshanski, *Point processes and the infinite symmetric group. Part I: The general formalism and the density function*, In: The orbit method in geometry and physics: in honor of A. A. Kirillov (C. Duval, L. Guieu, and V. Ovsienko, eds.), Birkhäuser Verlag, Basel, 2003, Progress in Math. 213.
- [35] O. Pâtu and A. Klümper, *Correlation lengths of the repulsive one-dimensional Bose gas*, Phys. Rev. A **88** (2013), 033623.
- [36] A. M. Polyakov, *Conformal symmetry of critical fluctuations*, JETP Lett. **12** (1970), 381.
- [37] K. Sakai, M. Shiroishi, J. Suzuki, and Y. Umeno, *Commuting quantum transfer matrix approach to intrinsic Fermion system: Correlation length of a spinless Fermion model*, Phys. Rev. B **60** (1999), 5186.
- [38] J. Sato, B. Aufgebauer, H. Boos, F. Göhmann, A. Klümper, M. Takahashi, and C. Trippé, *Computation of static Heisenberg-chain correlators: Control over length and temperature dependence*, Phys. Rev. Lett. **106** (2011), 257201.
- [39] A. Shashi, M. Panfil, J.-S. Caux, and A. Imambekov, *Exact prefactors in static and dynamic correlation functions of one-dimensional quantum integrable models: Applications to the Calogero-Sutherland, Lieb-Liniger, and XXZ models*, Phys. Rev. B **85** (2012), 155136.
- [40] A. Sommerfeld, *Zur Elektronentheorie der Metalle auf Grund der Fermischen Statistik*, Z. Phys. **47** (1928), 1.
- [41] M. Suzuki, *Transfer-matrix method and Monte Carlo simulation in quantum spin systems*, Phys. Rev. B **31** (1985), 2957.
- [42] M. Suzuki and M. Inoue, *The ST-transformation approach to analytic solutions of quantum systems. I. General formulations and basic limit theorems*, Prog. Theor. Phys. **78** (1987), 787.
- [43] M. Takahashi, *Correlation length and free energy of the  $S = \frac{1}{2}$  XYZ chain*, Phys. Rev. B **43** (1991), 5788.